A classification of cognitive tasks and operations of thought involved in addition and subtraction problems

In Addition and subtraction : a cognitive perspective
Hillsdale NJ, Lawrence Erlbaum

Carpenter T.P., Moser J.M., Romberg T.A. (Eds)
1982, pp.39-59

PSYCHOLOGY REVIVALS

Addition and Subtraction

A Cognitive Perspective

Edited by

Thomas P. Carpenter, James M. Moser and Thomas A. Romberg

REVIVALS

4 A Classification of Cognitive Tasks and Operations of Thought Involved in Addition and Subtraction Problems

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The purpose of this chapter is to present and explain a classification of additive relationships helpful in interpreting the procedures students use in solving addition and subtraction problems and in understanding more about the difficulties they meet. It also provides a framework for understanding the meaning of different symbolic representations of addition and subtraction, and a coherent basis for designing experiments on these mathematical processes. The theoretical reasons for the distinctions in the classification scheme are both psychological and mathematical. Let me illustrate with three examples of problems:

Problem A. There are 4 boys and 7 girls around the table. How many children are there altogether?

Problem B. John just spent 4 francs. He now has 7 francs in his pocket. How much did he have before?

Problem C. Robert played two games of marbles. On the first game, he lost 4 marbles. He played the second game. Altogether, he now has won 7 marbles. What happened in the second game?

Although a simple addition, 4 + 7, is needed in all three cases, Problem B is solved 1 or 2 years later than A, and C is failed by 75% of 11-year-old students. There must be some logical or mathematical difficulties in the last two problems that do not exist in the first.

To study this sort of hierarchy, I believe that a psychogenetic approach would be valuable. Until now psychogenetic theory has not been applied to specific content areas. Its use has been restricted to general problems of development, whereas study of specific content has been the object of learning theory. I consider this division to be somewhat misleading. It would be fruitful for education to consider a synthesis of psychogenesis and learning. One way to construct such a synthesis is to consider that knowledge is organized in "conceptual fields," the mastery of which develops over a long period of time through *experience, maturation,* and *learning*. By conceptual field, I mean an informal and heterogeneous set of problems, situations, concepts, relationships, structures, contents, and operations of thought, connected to one another and likely to be interwoven during the process of acquisition. For example, the concepts of multiplication, division, fraction, ratio, proportion, linear function, rational number, similarity, vector space, and dimensional analysis all belong to one single large conceptual field, the field of "multiplicative structures."

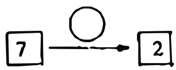
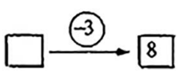
Similarly, I believe that the concepts of measure (of discrete sets and of other magnitudes), addition, subtraction, time transformation, comparison relationship, displacement and abscissa on an axis, and natural and directed number are also elements of one single conceptual field, the field of "additive structures." The progressive understanding of that field develops over a long period of time, from the age of 3 or 4 until at least age 15 or 16. The psychogenetic study of the acquisition of that field requires the analysis of the different relationships involved, and the hierarchical study of the different classes of problems that may be offered to students. It requires also the study of the different procedures and the different symbolic representations that students may use.

Children usually build up a conceptual field through experience in daily life and school. It is fruitful to plan and perform didactic experiments in order to understand more about how the field is constructed. For such experiments, it is essential to know which structures and classes of problems are the most easily understood by young students, which are the next, and so on. We must also know which procedures are the most naturally used by children or the most easily assimilated when taught. The same is true for symbolic representations: diagrams of different kinds, equations of different sorts. Such studies should enlighten our view of the slow process of acquisition and give us a better understanding of children's behavior.

To interpret the behavior of children faced with elementary arithmetic problems, I find it essential to distinguish two sorts of calculus: "relational calculus" and "numerical calculus." By "numerical calculus" I mean ordinary operations of addition, subtraction, multiplication, and division. By "relational calculus" I mean the operations of thought that are necessary to handle the relationships involved in the situation. This calculus can usually be expressed in terms of theorems, when it is valid, or in terms of false inferences, when it is not. But these theorems, assumptions, and inferences are not necessarily expressed or explained by children; they can only be hypothesized by observing children's actions. Let us call them "theorems in action" or "inferences in action."

If one wants to take such inferences into account, one has to develop a mathematical frame of reference of elementary arithmetic problems that includes aspects of the problem situations that are not usually taken into account by mathematicians or by text authors. For example mathematicians are not in-

TABLE 4.1
Relational Calculus and Numerical Calculus

Illustrative Examples		Diagram and Relational Calculus	Numerical Calculus
F+	Fred had 3 sweets, he buys 14 sweets. How many sweets does he have now?	 apply a direct positive transformation to the initial state	addition
F−	Fred had 17 sweets, he eats 4 of them. How many sweets does he have now?	 apply a direct negative transformation to the initial state	subtraction
T+	Tony had 5 marbles. He plays with Robert. He has now 12 marbles. What has happened during the game?	 find the difference between two states I < F	subtraction
T−	Tony had 7 marbles. He plays with Robert. He now has 2 marbles. What has happened during the game?	 find the difference between two states I > F	subtraction
I+	Inge has just received 4 dollars from her mother. She now has 7 dollars. What did she have before?	 find the inverse of a positive transformation and apply it to the final state.	subtraction
I−	Inge has just spent 3 dollars to buy sweets. She has now 8 dollars. What did she have before?	 find the inverse of a negative transformation and apply it to the final state.	addition

terested in the concepts of time and dimension. But children take time aspects and dimensional aspects into consideration. One can hardly understand what they are doing if one keeps these aspects out of the frame of reference. To illustrate, let us take Problem D and Problem E:

> Problem D: 4 boys, 7 girls; how many children are there altogether?

This exemplifies the classical measure–measure–measure relationship: One measure is the composition of two elementary measures. There are only two classes of such problems: first, find the sum knowing the elementary measures, and second, find one elementary measure knowing the sum and the other elementary measure. These two classes of problems are in one-to-one correspondence with the numerical operations of addition and subtraction.

> Problem E: John has just spent 4 francs, he now has 7 francs; how much did he have before?

This exemplifies a different relationship, a measure–transformation–measure relationship. From these examples the six distinct classes of problems can be generated (Table 4.1). However, these six classes of problems are not in a simple correspondence with the numerical operations of addition and subtraction.

If problem solving is both the source and the basis of operational knowledge, it is essential to use a classification that encompasses all (or most) classes of problems and most aspects of problem solving, rather than a framework based only on the numerical operations (addition, subtraction) and on the concepts of number and equation, which fail to map many relevant features of the cognitive tasks involved. This does not mean that the concepts of number and equation are not essential. However, they are abstract entities that elementary school children find difficult to handle. The reason is that they are a condensation of too many different relationships and entities.

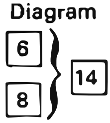
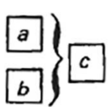
	Diagram	Equation
Problem F		$6 + 8 = 14$
	Squares symbolize measures.	The sign + is the symbol for addition of natural numbers (internal operation in N).
		There are two main classes of problems: —find c, knowing a and b —find a (or b), knowing b (or a) and c.

FIG. 4.1. Representation of Category I: Composition of two measures.

BASIC CATEGORIES OF RELATIONSHIPS

To represent the six main categories of relationships that I have found necessary to distinguish, let me use special symbols and notation. The six categories are explained in Fig. 4.1 through 4.6 and the notation is summarized in Fig. 4.7.
Category I: Composition of two measures. Problem F illustrates this category:

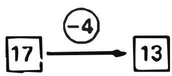
> Problem F: Peter has 6 marbles in his right-hand pocket and 8 marbles in his left-hand pocket. He has 14 marbles altogether.

The two classes of problems in this category are mentioned in Fig. 4.1.
Category II: A transformation links two measures (state-transformation-state [STS]). Problem G illustrates this category:

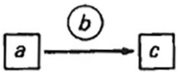
> Problem G: Peter had 17 marbles before playing. He has lost 4 marbles. He now has 13 marbles.

Diagram

Problem G



The horizontal arrow symbolizes a transformation linking a measure to another one.



Equation

$17 + (-4) = 13$

The sign $+$ is the symbol for addition of a natural number and a directed number (external operation of Z on N).

There are six main classes of problems:

—find c, knowing a and b $\begin{cases} b>0 & \text{F+} \\ b<0 & \text{F−} \end{cases}$

—find b, knowing a and c $\begin{cases} c>a & \text{T+} \\ c<a & \text{T−} \end{cases}$

—find a, knowing b and c $\begin{cases} b>0 & \text{I+} \\ b<0 & \text{I−} \end{cases}$

FIG. 4.2. Representation of Category II: State-Transformation-State (STS).

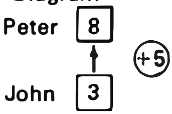
The six classes of problems (F+, F−, T+, T−, I+, I−) outlined in Table 4.1 and in Fig. 4.2 are in this category. Category II refers to the concept of a transformation happening in time.
Category III: A static relationship links two measures (state-relationship-state [SRS]). Problem H illustrates this category:

> Problem H: Peter has 8 marbles. He has 5 more than John. John has 3 marbles.

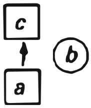
Figure 4.3 represents this category, which has also six classes of problems, analogous to those of Category II. I have found it necessary to distinguish this category from Category II to highlight the difference between dynamic transformations and static relationships.

Diagram

Problem H



The vertical arrow symbolizes a static relationship (a comparison in this case) linking two measures.



Equation

$3 + (+5) = 8$

The sign $+$ is the symbol for addition of a natural number and a directed number as in category II.

There are six main classes of problems (analogous to those described in Fig. 4.2).

FIG. 4.3. Representation of Category III: State–Relationship–State (SRS).

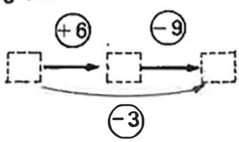
Category IV: Composition of two transformations (transformation–transformation–transformation [TTT]). Problem I illustrates this category.

> Problem I: Peter won 6 marbles in the morning. He lost 9 marbles in the afternoon. Altogether he lost 3 marbles.

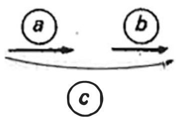
This problem is diagrammed in Fig. 4.4. There are many classes and subclasses of problems in Category IV. To generate these classes see Fig. 4.4.

Diagram

Problem I



The lower arrow symbolizes the composition of the two upper ones. The states are not known.



Equation

$(+6) \oplus (-9) = (-3)$

The sign $\oplus$ is the symbol for addition of two directed numbers (internal operation in Z).

There are three main classes of problems:
- —find c, knowing a and b
- —find b, knowing a and c
- —find a, knowing b and c

Within each of these main classes, there are subclasses depending on the values of the data. For instance, if a and c are given, one gets the following cases:

	$a>0$, $c>0$	$a<0$, $c<0$	$a>0$, $c<0$	$a<0$, $c>0$
$\|a\| < \|c\|$				
$\|a\| > \|c\|$				

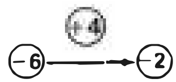
FIG. 4.4. Representation of Category IV: Composition of two transformations (TTT).

Category V: A transformation links two static relationships (relationship-transformation-relationship [RTR]). Problem J illustrates this category.

> Problem J: Peter owed Henry 6 marbles. He gives him 4. He still owes Henry 2 marbles.

Different diagrams and equations are possible in this case (see Fig. 4.5).

Diagram	Equation
Problem J	

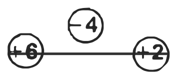


$(-6) \overset{\rightarrow}{+} (+4) = (-2)$

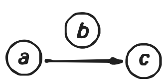
States represent Peter's account in Henry's book of credit.

The sign $\overset{\rightarrow}{+}$ is the symbol for addition of two directed numbers. It actually stands in this case for an external operation of transformations on states.

Another diagram could be:



States would then represent Henry's account in Peter's book.



There are three main classes of problems as in Category IV. Each one can also be subdivided into different subclasses.

FIG. 4.5. Representation of Category V· Relationship-Transformation-Relationship (RTR).

Category VI: Composition of two static relationships (relationship-relationship-relationship [RRR]). Two problems will illustrate this category.

> Problem K: Peter owes 8 marbles to Henry, but Henry owes 6 to Peter. So Peter owes 2 marbles to Henry.
>
> Problem L: Robert has 7 marbles more than Susan. Susan has 3 marbles less than Connie. Robert has 4 marbles more than Connie.

This category is represented in Fig. 4.6. Because there is no fixed time order, there are several possible representations of the same situation.

The classification of problems into these six main categories is comprehensive. This classification relies on the distinction between three main concepts; measure, time transformation, and static relationship.

In this regard I would like to stress two points: First, time transformations and static relationships are not adequately represented by natural numbers, and problems that involve them are not adequately represented by equations in *N*. *N* is adequate for measures (of discrete sets) and for Category I relationships. It is not

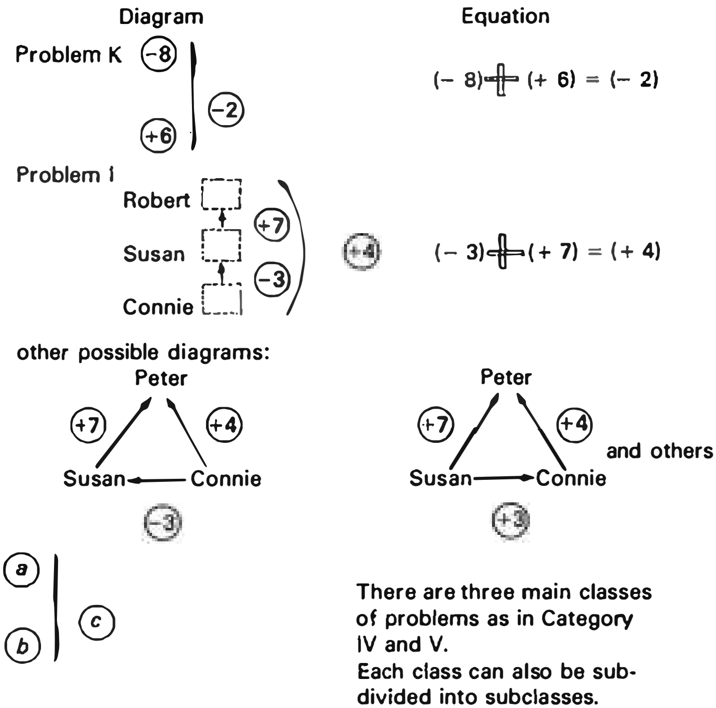


FIG. 4.6. Representation of Category VI: Relationship - Relationship - Relationship (RRR).

suitable for Category II, III, IV, V, or VI relationships because they involve elements that should be represented by directed numbers. However, students meet these categories long before they learn about directed numbers. Thus, there is a discrepancy between the structure of problems that children meet and the mathematical concepts that they are taught.

Second, the term "dynamic" is ambiguous. It may refer to the action of an actor or it may refer to a transformation that has nothing to do with any action. Initially, I was interested in the idea that because action plays a central role in building concepts, teaching children a dynamic model would be more productive than teaching a static one. My present view is more careful: One must distinguish carefully between the concept of *action*, the concept of *transformation*, and the concept of *operation*. Where action refers to an actor's doing, transformation refers to a change in the state of nature, and operation refers to the procedure used to solve a problem. Let us take three examples:

> John had 4 sweets. He buys 3 sweets. How many sweets does he have now?

If a student solves the problem by saying, ‘‘4 + 3 = 7,’’ then ‘‘+3’’ stands for John’s action, or the transformation of John’s collection, or Peter’s operation to solve the problem.

> John had 4 sweets. His mother gives him 3 sweets. How many sweets does he have now?

Now if the student says, ‘‘4 + 3 = 7,’’ then ‘‘+3’’ may stand for the transformation of John’s collection or Peter’s operation to solve the problem.

> John had 4 sweets. He has just eaten 3 sweets. How many did he have before?

Now if the student says, ‘‘4 + 3 = 7,’’ then ‘‘+3’’ stands for Peter’s operation to solve the problem. It does not represent John’s action or the transformation of John’s collection.

Suppose the student writes ‘‘4 + 3 = □,’’ or ‘‘4 $\xrightarrow{+3}$ □.’’ For the first two problems, has he or she written an equation or a diagram of the problem, or has he or she just written down his or her procedure to solve the problem? It is impossible to decide. For the third problem, if the student writes ‘‘4 + 3 = □,’’ or ‘‘4 +3 □,’’ he or she has clearly written down his or her procedure and not an equation or a diagram. To depict these, he or she should have written ‘‘□ − 3 = 4,’’ or □ $\xrightarrow{-3}$ 4.’’ Thus, for certain classes of problems it is possible to decide whether a symbolic expression represents the actual situation, or the procedure used to solve the problem.

Hence, it should be clear that the term ‘‘dynamic’’ is ambiguous because it refers to different concepts that may be or may not be congruent. The hypothesis that dynamic situations are easier than static ones may be true for some situations, but not for all, as is shown in the next section.

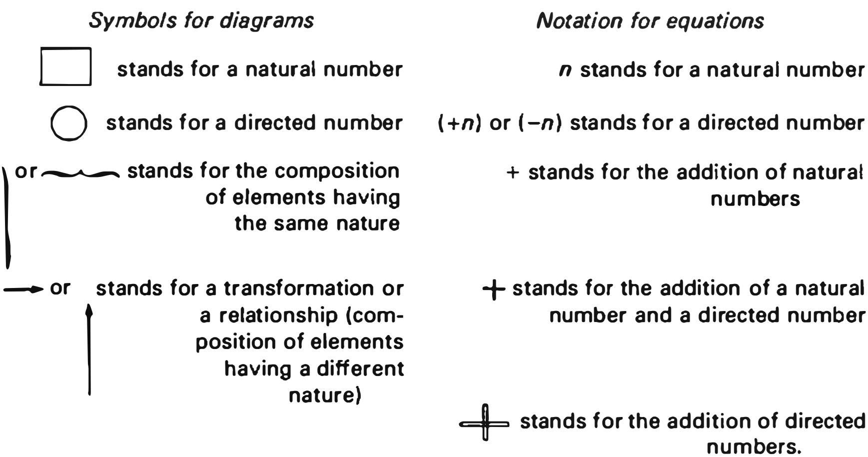


FIG. 4.7. Symbols and notation used to represent categories of addition and subtraction problems.

EXPERIMENTAL RESULTS

Experiment 1. In our first experiment (Vergnaud & Durand, 1976) we examined differences between Category II and Category IV problems and also differences between distinct classes of problems in Category IV. Because we stressed the relational aspects, small numbers were used. We used pairs of problems needing the same addition or subtraction equation but having different structures.

Table 4.2 shows pairs of problems and the number of correct responses for 28 children at each of five grades. There are some very clear differences. A 1-year decalage appears for the pair of problems Pierre and Paul, and at least a 3-year decalage for the pair Bertrand and Bruno. The decalage is smaller for the pair Claude and Christian. Although the same numerical calculus is needed in each pair of problems, the composition of transformations (Category IV) is more difficult than the application of transformations to states (Category II). However, the differences are not homogeneous for the different cases. For instance, the small difference between Christian and Claude may come from the fact that both cases can very easily be mapped into the composition-of-measures model, all transformations being positive. This is not the case with Pierre and Paul, nor with Bertrand and Bruno. The graphs show clearly the gap between adding two transformations and applying a transformation to a state. The gap is still bigger between finding the difference of two transformations having opposite signs and inversing a direct negative transformation.

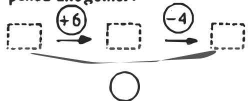
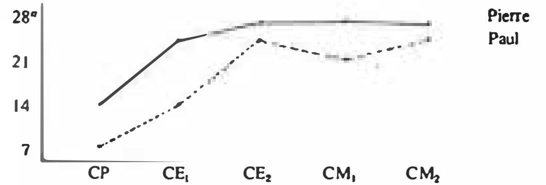
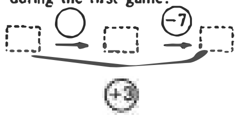
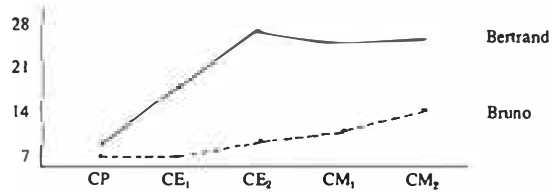
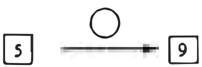
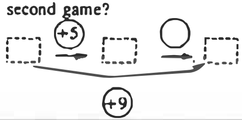
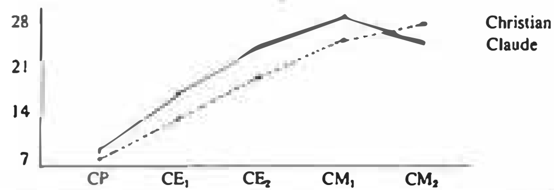
Also, if one compares the problems in Category II, one finds some interesting differences. For instance, it is easier, in the case of a direct negative transformation, to calculate the final state (Pierre) than the initial state (Bertrand). There is roughly a 1-year decalage.

We also studied different Category IV problems, implying two successive transformations (games of marbles) and the composed transformation.

Starting from the TTT model shown in Fig. 4.4, we studied the class of problems illustrated by problem "Christian" (find b, knowing a and c or find T_2 knowing T_1 and T_3) and chose five problems out of the different possible cases shown in Table 4.3. The problem types again are identified in Table 4.3 by the names of the actors in each type. As we expected, Christian was found easier than Jacques, Jacques easier than Didier, and Christian easier than Didier. Cross-tabulations for those three situations are shown in Table 4.4. Further, Olivier and Vincent were particularly difficult. They were failed by 80% of the students of the last primary school grade.

The actual procedures used by students to solve these problblems are not just the canonical relational calculus. For instance, in Claude (Table 4.2) (find T, knowing the initial and final states when T is positive), we found not only a *difference* procedure (if T links two states, its value is the difference), but also a *complement* procedure (which consists of finding directly what should be added to the initial state to reach the final state).

TABLE 4.2
Comparison of Category II and Category IV Problems

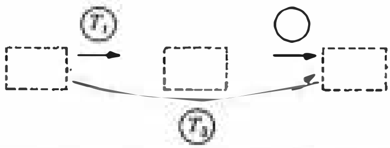
Category II (STS) *State-Transformation-State*	*Category IV (TTT)* *Transformation-Transformation-Transformation*	
Pierre has 6 marbles. He plays one game and loses 4 marbles. How many marbles does he have after the game? 	*Paul* plays two games of marbles. At the first game, he wins 6 marbles. At the second game he loses 4 marbles. What has happened altogether? 	
Bertrand plays a game of marbles. He loses 7 marbles. After the game he has 3 marbles. How many marbles did he have before the game? 	*Bruno* plays two games of marbles. He plays a first game, then a second game. At the second game he loses 7 marbles. After those two games he has won 3 marbles altogether. What has happened during the first game? 	
Claude has 5 marbles. He plays one game. After the game he has 9 marbles. What has happened during the game? 	*Christian* plays two games of marbles. At the first game he wins 5 marbles. He plays a second game. After these two games he has won 9 marbles altogether. What has happened during the second game? 	

CP: first grade of French primary school (6-year-olds); CE_1, CE_2, CM_1, CM_2 are next grades.
[a]number correct

TABLE 4.3
TTT Five Problem Cases Studied in Vergnaud and Durand, 1976

	$T_1 > 0$, $T_3 > 0$	$T_1 < 0$, $T_3 < 0$	$T_1 > 0$, $T_3 < 0$	$T_1 < 0$, $T_3 > 0$
$\lvert T_3\rvert > \lvert T_1\rvert$	Christian	Jacques	Olivier	
$\lvert T_3\rvert < \lvert T_1\rvert$		Didier	Vincent	

Example: Christian plays two games of marbles. In the first game he wins 5 marbles ($T_1 > 0$). He plays a second game. After these two games he has won 9 marbles altogether ($T_3 > 0$ and $\lvert T_3\rvert>\lvert T_1\rvert$). What has happened during the second game?



Similarly, in Bertrand (Table 4.2) (find the initial state, knowing T and the final state, when T is negative), we found not only the *inversion* procedure (take T^{-1} and apply it to the final state) but also the *hypothetical-initial-state* procedure (which is to make a hypothesis on the initial state, apply the direct transformation, find the outcome, compare with the actual final state, and either correct the hypothesis accordingly or make a new hypothesis). The *complement*

TABLE 4.4
Cross-Tabulation for Three Problem Pairs for the Category IV (TTT) Problem Cases

Christian × Jacques

		Jacques C	Jacques I	n
Christian	C	72	19	91
	I	5	44	49
	n	77	63	140

Jacques × Didier

		Didier C	Didier I	n
Jacques	C	49	26	75
	I	6	31	37
	n	55	57	112

Christian × Didier

		Didier C	Didier I	n
Christian	C	53	36	89
	I	2	21	23
	n	55	57	112

C = Correct
I = Incorret

procedure can be successfully used when T is positive. However for this case it leads to a failure.

In Category IV problems, *hypothesis* procedures are very often used because they require no inversion. Also, very often children interpret transformations as states. Sometimes, this interpretation enables the student to handle the problem, but often it is completely misleading.

Experiment 2. Conne (1979) repeated the first experiment and found essentially the same results. He did a detailed analysis of students' procedures and found that many procedures and explanations produced by children can be interpreted in terms of a functional model involving "what do I start from; what do I do next (add or subtract)?" This model is not suitable for all problems, but, in many cases it enables students to approach problems they would otherwise miss. Of course, using the functional model requires the identification of the starting point and the action.

For example, in T_3 problems (find T_3 knowing T_1 and T_2) the canonical solution requires the composition of two transformations. In fact, students very often consider that T_1 is the initial state and that T_2 operates on T_1. This interpretation can lead to success when T_1 and T_2 are positive, or when T_1 is positive and T_2 is negative and smaller in absolute value than T_1. But this model is not operational in other cases. In fact children's answers can be interpreted as inferences concerning the state of affairs before or after the games. For example: "wins 6 marbles" is interpreted as "he has 6 marbles," and "loses 7 marbles" is interpreted as "he had 7 marbles and he has no marbles left."

In problem Paul (Table 4.2) the answer is often given as the value of the final state: "Two marbles are left." In Vincent (Table 4.3), T_3 is often considered as operating on T_1: "Six marbles are left." This interpretation depends on the difficulty of the numerical values chosen. The same text may be interpreted in terms of transformations (correct) when the numerical values are easily handled (problems Christian and Jacques, Table 4.3), for instance, in which T_1 and T_3 have the same sign and ($|T_3| > |T_1|$), and in terms of states (incorrect) when the numerical values are difficult to deal with (Vincent, Table 4.3, for instance, in which T_1 and T_3 have opposite signs).

The functional model is not universal. When children map by forcing the problem into the functional model, they may reverse the order of transformations or even change a loss into a gain and vice versa, so that the problem suits the model.

Conne also compared T_2 problems (find T_2 knowing T_1 and T_3) and T_3 problems (find T_3) and found several interesting differences. First, the variety of answers is smaller for T_2 problems than for T_3. Second, students are more inclined to combine data and make a calculus (usually a subtraction) for T_2 problems. Finally, when students' answers consist of repeating data, they usually repeat only one datum for T_2 problems and both data for T_3 problems.

Marthe (1979) obtained more evidence in experimenting with older students (secondary school, 11-year-olds to 15-year-olds). He concentrated his attention on Category IV and V problems that can be represented by the equation $a + x = b$ where a, b, and x are directed numbers. He studied two cases: when a and b had opposite signs ($a > 0$, $b < 0$, and $|a| > |b|$), and when a and b had the same sign ($a > 0, b > 0$ and $|a| > |b|$).

His results show that the "opposite-sign" problems are always more difficult than the "same-sign" problems. For same-sign problems, Category IV was very difficult, even more difficult than the Category VI problems. There is an increase in performance across grades. However, the majority of students at the age of 15 still perform poorly. This indicates that the growth of understanding of the conceptual field of additive structures takes a long time and is not completed by the age of 15.

Experiment 3. Fisher (1979) compared different problems requiring subtraction. He worked with second-grade students (7- and 9-year-olds) during the time they learn subtraction. He used three classes of problems and different numerical data in each case.

Fisher wanted to know whether these three classes of problems were hierarchically ordered and how learning subtraction would modify students' behavior. His results showed that Category II final-state problems, in which students have to find out what the final state is knowing the initial state and the transformation are much easier than others. Performance is very low for most other problems. The order of the data is not a relevant (or important) factor. Instead, the most relevant factor is the structure of the problem.

Fisher's main conclusion is that subtraction is first understood as a direct negative transformation, not as the inverse of addition of measures, nor as the inverse of a direct positive transformation. This is very important because it has to do with the "dynamic" versus "static" problem and the relationship between addition and subtraction. If the very first model of subtraction for the child is the Category II structure, one must specify that Fisher's conclusion is true only for the "find-the-final-state" paradigm. Subtraction is not merely the inverse of addition. It has its own meaning as a direct transformation of the state of nature.

Fisher also tested the level of students on the class inclusion test and found that intermediary level 2b (just before the last level, according to Piagetian description) was a necessary condition for success on Category I subtraction problem. After six months, the order in which the ability to solve various classes of problems was acquired was in accordance with the hypothesized hierarchy. Furthermore, progress was better on Category I problems for students who correctly solved Category II final-state problems at the beginning of the school year. The most frequent error observed was giving one of the numerical data as the answer.

ARE SYMBOLIC REPRESENTATIONS USEFUL?

Although the problem of representation has been raised before in this chapter, it has not been discussed. My use of diagrams was intended to express clearly the differences my analysis takes into account. Different symbols for different sorts of addition were used in the same way. The question is: *How useful are symbolic representations?* This is not a rhetorical question. When children solve a problem, they often make the calculations first and write the symbolic representation, whatever it is, afterwards. When one studies the capacity of students to solve subtraction problems either in the form of a ready-made arrow diagram or in the form of a verbal problem, one does not find much difference. Yet, it is hard to conceive that symbolic representations are useless. Mathematicians have devoted long hard discussions to the problem of symbolization and we spontaneously use symbols to make other people understand what we are talking about. I would like to propose two criteria for the efficiency of symbolic representations.

> First criterion: *Symbolic representations should help students to solve problems that they would otherwise fail to solve.*

This criterion is not easily met. We must make an effort to imagine problems for which it may be verified. My suggestion is that symbolic representations may be helpful when there are many data and when there are different structures.

> Second criterion: *Symbolic representations should help students in differentiating various structures and classes of problems.*

This criterion is nearly a direct consequence of the first. Even if symbolic representations are not used to solve problems, they may be useful in helping students to analyze and differentiate structures.

Whereas the first criterion is an immediate or short-term criterion, the second one should permit a long-term evaluation. Of course, these criteria should be used to evaluate different sorts of symbolic systems, at different stages of the acquisition of additive structures. For instance, Euler–Venn diagrams may be helpful for certain classes of problems at the beginning of primary school and equations may be more helpful at the end of secondary school. A similar hypothesis may be made for arrow diagrams and for other symbolic representations.

Thus, we should take time to examine different symbolic systems and see what they can symbolize correctly, what limits they have, and what their advantages and inconveniences are. To illustrate this question let us look at two problems and five different representations for each of them (see Table 4.5).

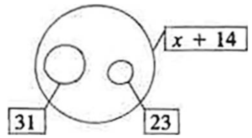
It can be seen from Table 4.5 that Euler–Venn diagrams and distance diagrams do not discriminate problems A and B. Algebraic equations are ambiguous

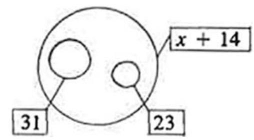
TABLE 4.5
Five Symbolic Representations for Two Problems

A. Peter played marbles with friends. In the morning, he won 14 marbles. In the afternoon he lost 31 marbles. He now has 23 marbles. How many marbles did he have before playing?

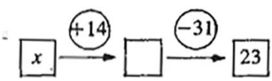
B. In the morning, Peter had 14 marbles. He played with John in the morning and with Tony in the afternoon. In the afternoon, he lost 31 marbles. He now has 23 marbles. What happened in the morning when he played with John?

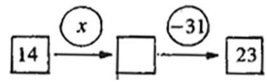
1. Euler–Venn Diagrams





2. Transformation Diagrams



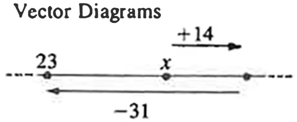


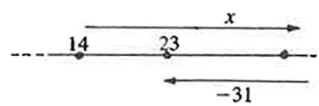
3. Algebraic Equations

A: $x + 14 - 31 = 23$

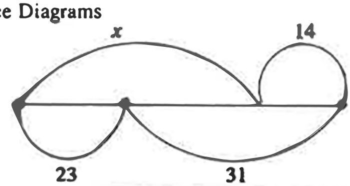
B: $14 + x - 31 = 23$

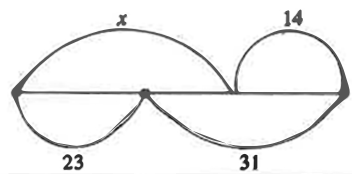
4. Vector Diagrams





5. Distance Diagrams





because $x + 14$ in problem A and $14 + x$ in problem B do not mean the same thing (if x were negative in problem B, it would still be written $14 + x$). Transformation diagrams and vector diagrams discriminate A and B but, before representing problem B by vectors, you have to think of the sign of the transformation x to know which way it goes. In other words, vector diagrams might suppose that you have already solved the problem. It seems to me that Euler–Venn diagrams and distance diagrams also suppose that the problem has already been solved. The only way to decide is to make experiments.

I wish I could show clear empirical results to sustain my theoretical views. This is not possible. I can mention only one experiment. I do it very briefly, because it does not concern the initial learning of additive structures. This experiment was made with 11- to 13-year-old students (last grade of primary school—first grade of secondary school) on a variety of Category I, II, III, and IV problems for which children had to deal with more than two data and more than one structure. Two versions in natural language had been written for each structure (see structures in Fig. 4.8). We compared the distributions of the procedures used by subjects for pretest and posttest sessions. Between the test sessions, students had worked 5 weeks with other problems, equations, and arrow diagrams. Table 4.6 shows results concerning the use of equations and the use of diagrams. The results can be summarized as follows:

Place of the unknown. Suppose a student writes $x =$ ____ or ____ $= x$. In these instances, the other member of the equation is usually nothing else but the sequence of numerical operations needed to calculate x. In most cases, as in the five structures, the unknown x should appear as an element inside one member of the equation.

Chaining equalities. Fairly often, students are able to make a correct response but they write a wrong statement because they chain equalities that should be kept separate. For instance,

Step 1: $1063 + 217 = 1280$.
Step 2: Instead of writing on the next line
$1280 - 425 = 855$,
they pursue on the same line
$1063 + 217 = 1280 - 425 = 855$.

This error clearly indicates that for many students the equality sign does not stand for a relationship between numbers but for a procedure to calculate the unknown.

Inverse writing of subtraction. Writing the sentence $425 - 1280 = 855$ instead of $1280 - 425 = 855$, is a well-known error. It means that the student is not interested in the sentence, only in the result. The sign "−" represents the

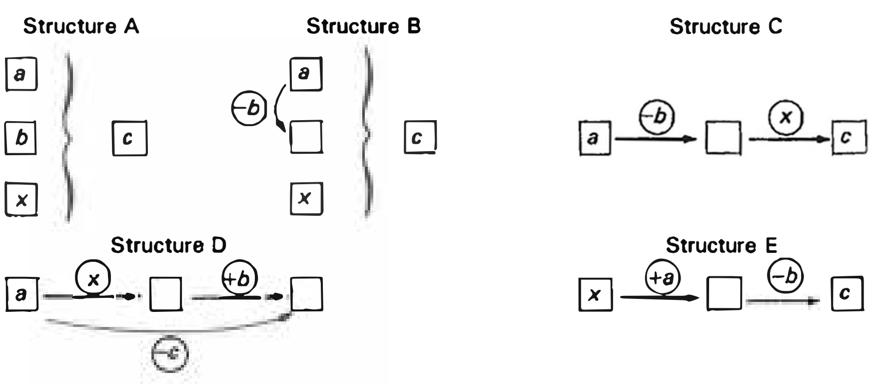


FIG. 4.8. Five structures used to study use of diagrams.

TABLE 4.6
The Use of Equations and Diagrams on a Set of Problems

Equations
Unknown placed inside an expression

	CM_2	6 éme	Total
Pretest	1	1	2
Posttest	8	8	16

Chaining equalities (error)

	CM_2	6 éme	Total
Pretest	7	7	14
Posttest	7	6	13

Inverse writing of subtraction (error)

	CM_2	6 éme	Total
Pretest	19	5	24
Posttest	8	12	20

Diagrams
Correct use of diagrams

	CM_2	6 éme	Total
Pretest	1	0	1
Posttest	31	8	39

CM_2: 50 subjects–last grade of elementary school (11 years)
6 éme: 50 subjects–first grade of secondary school (12 years)
Total: 100 subjects

difference between two numbers, regardless of which the big one and the small one are. What matters to the student is not the problem of writing a correct relationship but the problem of finding the result.

Both chaining equalities and inverse writing of subtraction are understandable. However, they undoubtedly mean that equations and equalities are not used to extract and represent relevant relationships between numbers but rather to recall the sequence of numerical operations used to calculate a result. Table 4.6 shows that only a few students (2% at the pretest and 16% at the posttest) are able to place the unknown inside an expression at least once in the test. It shows also that the two classical errors (chaining equalities and inverse writing of subtraction) are frequent.

Diagrams can be considered a kind of equation with additional information specified (measures, states, transformations, or relationships). So the problem is to determine whether they can be used more easily by children to represent problems. Table 4.6 shows that although students were not acquainted with diagrams at the pretest, almost 40% of them were able to use diagrams at the posttest. This makes credible the thesis that, at this stage of development (11–12 years), diagrams are more appropriate than equations. But we still have to prove it.

CONCLUSION

In the last part of this chapter, the term *representation* has been used with the restrictive meaning of "symbolic system": a set of signs, syntax, or operations on the elements of the system. Earlier I referred to different categories that are essentially conceptual. Concepts and symbols are two sides of the same coin and one should always take care to view students' use of symbols in the light of their use of concepts. In other words, the ability to solve problems in natural language, issued from ordinary social, technical, or economical life, is the best criterion of the acquisition of concepts. Reciprocally, it is essential to know how mathematical symbolization helps students. My view is that the acquisition process consists of building up relational invariants, analyzing their properties, and building up new relational invariants. Some quantitative invariants have been studied by Piaget within the context of conservation experiments. Little work has been done on relational invariants. We should do more, because arithmetic, geometry, physics, and other fields of knowledge consist essentially of relational invariants.

In this chapter, I have dealt with relationships that involve time and have shown that time is the source of important differences among the problems children try to solve. Another source of differences is the inclusion relationship. Carpenter and Moser (1979) have raised this problem under the "part-part-whole" category, and I have also mentioned briefly some results, obtained by Fisher (1979), that show the relevance of this aspect.

Looking at the three main concepts I have used in my classification (measure, time transformation, and static relationship), and at Categories I, II, and III, one can see that there are three criteria involved in these distinctions. These criteria are not independent from one another, although each of them carries some separate information.

> The first criterion is: Either all elements are measures (Category I) or one element is not a measure (Categories II and III).
> The second criterion is: Either time is involved (Category II) or time is not involved (Categories I and III).
> The third criterion is: Either there is an inclusion relationship (Categories I and II) or there is no inclusion relationship (Category III).

I have not emphasized this last criterion in this chapter. It is probably very important because Categories I and II both convey an inclusion relationship. In Category I, the elementary sets (or magnitudes) are parts of the whole; in Category II, either the initial set (or magnitude) is part of the final one, or the final one is part of the initial one. In Category III, because the two linked measures are simultaneously present, there is not necessarily (and usually there is not) any inclusion relationship. For example in "Peter has five more marbles than John," none of the quantities involved (Peter's marbles, John's marbles, or the difference) is included in any other set. Little experimental work has been done on these aspects, but I would expect it to be fruitful.

Additive structures are a difficult conceptual field, more difficult than most mathematics teachers expect. Understanding additive structures is a long-term process that starts with some simple find-the-final-state problems and goes on into adolescence with subtraction of opposite-sign transformations.

Some problems are more easily solved than others; likewise, children find some procedures more natural than others. We should use these procedures in teaching, even if they are not canonical but only local procedures, because they are probably the best way to help children build up the canonical ones. Brousseau (1978) has developed a systematic theory of how to get children to a new step, which concerns situations in which children have to act, to formulate, and to validate. For instance, changing the numerical values is very often necessary to get children to move from a local procedure to a more powerful canonical one.

What is true for problems and procedures is also true for symbolic representations; some representations such as equations are more powerful than arrow diagrams or Euler-Venn diagrams. However, these equations should represent meaningful situations. Consequently, the problem of knowing which intermediate representations may help students to handle these situations is very important. Probably equations using directed numbers are too abstract, having too many different meanings mapped onto the same sign. For instance, the minus sign may stand for a direct negative transformation, a complement, the inverse of a positive transformation or relationship, a subtraction of two transformations, and so on. Equations using natural numbers are undoubtedly easier, but either they stand for the procedure used by students and not for the objective structure of problems, or they represent correctly the structure of only a few classes of problems.

My last comments are on the concept of "theorem in action." In Piaget's experiments, obviousness is the strongest criterion of cognitive appropriation. When children do not conserve, they find it obvious that things change; when they do conserve they find it obvious that things do not change. Similarly, when children solve an additive problem, they usually find it obvious that they should add or subtract. The relational calculus they have performed, implicitly, makes it clear that they should do a particular numerical operation. This is what I call a "theorem in action." In more sophisticated terms, understanding a new relational invariant or a new property of an invariant provides the choice of the right numerical operation. The child is now able to "arithmetize" a qualitative structure.

In conclusion, I believe that didactic and psychological research in mathematics teaching should pay more attention to the following four questions:

1. What are the easiest "theorems in action" used by students in solving verbal problems?
2. How should we get students to build up new theorems by presenting them with new situations?
3. How and with the help of which symbolic systems should we help students make these theorems explicit?
4. How can we make sure that theoretical theorems actually become theorems in action?

Practice and theory are sides of the same coin, and problem solving is certainly the source and the criterion of appropriated knowledge. The natural order of appropriation, when it exists and when it can be found, will be of great value in the classroom.

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