Multiplicative structures

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ACQUISITION OF MATHEMATICS CONCEPTS AND PROCESSES

Edited by

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CHAPTER 5

Multiplicative Structures

*Gerard Vergnaud**

History teaches us that science and technology have developed with the aim of solving problems. One of the most challenging points in education is probably to use meaningful problems so that knowledge, both in its theoretical and its practical aspects, may be viewed by students as a genuine help in solving real problems. However, this condition, that knowledge be both operational and interesting, cannot easily be satisfied.

Piaget has demonstrated that knowledge and intelligence develop over a long period of time, but he has done this by analyzing children's development in terms of general capacities of intelligence, mainly logical, without paying enough attention to specific contents of knowledge. It is the need to understand better the acquisition and development of specific knowledge and skills, in relation to situations and problems, that has led me to introduce the framework of conceptual fields. A *conceptual field* is a set of problems and situations for the treatment of which concepts, procedures, and representations of different but narrowly interconnected types are necessary.

Why is such a framework necessary?

1. It is difficult and sometimes absurd to study separately the acquisition of interconnected concepts. In the case of multiplicative structures, for instance, as we see in this chapter, it would be misleading to separate studies on multiplication, division, fraction, ratio, rational number, linear and n-linear function, dimensional analysis, and vector space; they are not mathematically independent of one another, and they are all present simultaneously in the very first problems that students meet.

*Other members of the research group whose experiments were reported are A. Rouchier, G. Ricco, P. Marthe, C. Landré, A. Viala, R. Metregiste and, recently, J. Rogalski and R. Samurcay.

2. It is also wise, in a psychogenetic approach to the acquisition of specific ideas, to delineate rather large domains of knowledge, covering a large diversity of situations and different kinds and levels of analysis. This enables one to study their development in the student's mind over a long period of time.

3. Finally, there are usually different procedures and conceptions and also different symbolic representations involved in the mastery by students of the same class of problems. Even though some of these conceptions and representations are weak or partially wrong, they may be useful for the solution of elementary subclasses of problems and for the emergence of stronger and more nearly universal solutions.

The conceptual-field framework makes it possible to study the organization of these interconnected ideas, conceptualizations, and representations over a period of time long enough to make the psychogenetic approach meaningful.

I have been interested in two main conceptual fields, additive structures and multiplicative structures, viewing them as sets of problems involving arithmetical operations and notions of the additive type (such as addition, subtraction, difference, interval, translation) or the multiplicative type (such as multiplication, division, fraction, ratio, similarity). Of course, multiplicative structures rely partly on additive structures; but they also have their own intrinsic organization which is not reducible to additive aspects. See Vergnaud, 1981; Vergnaud, 1982; Vergnaud, 1983 for additive structures.

Other important conceptual fields, interfering with these two, include (*a*) displacements and spatial transformations; (*b*) classifications of discrete objects and features, and Boolean operations; (*c*) movements and relationships among time, speed, distance, acceleration, and force; (*d*) parenthood relationships; and (*e*) measurement of continuous spatial and physical quantities.

Some of these last conceptual fields play an important part in the meaning and understanding of additive and multiplicative structures; reciprocally, the development of additive or multiplicative structures is necessary for the mastery of certain relationships involved in other conceptual fields. It is a complex landscape. Still, I find it fruitful to delineate distinct domains, if they can be consistently described, even though these domains are not independent.

Preliminary Analysis

Looking at multiplicative structures as a set of problems, I have identified three different subtypes: (*a*) isomorphism of measures, (*b*) product of measures, and (*c*) multiple proportion other than product.

Isomorphism of Measures

The isomorphism of measures is a structure that consists of a simple direct proportion between two measure-spaces M_1 and M_2. It describes a large number of situations in ordinary and technical life. These include: equal sharing (persons and objects), constant price (goods and cost), uniform speed or constant average speed (durations and distances), constant density on a line (trees and distances); on a surface, or in a volume. Four main subclasses of problems can be identified:

MULTIPLICATION

Schema 5.1 illustrates the isomorphism of measures for multiplication.

M_1	M_2
1	a
b	x

SCHEMA 5.1.

Example 1. Richard buys 4 cakes priced at 15 cents each. How much does he have to pay?

$a = 15, \quad b = 4, \quad M_1 = $ [numbers of cakes], $\quad M_2 = $ [costs].

Example 2. A farm of 45.8 ha produces 6850 kg of corn per ha. What will be the yield?

$a = 6850, \quad b = 45.8, \quad M_1 = $ [areas], $\quad M_2 = $ [weights of corn].

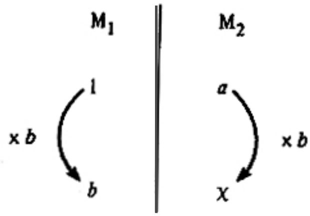
Multiplication problems do not consist of a three-term relationship but of a four-term relationship from which children have to extract a three-term relationship. They can do it by extracting either a binary law of composition or a unary operation. Each method implies different operations of thinking, as shown below.

Binary Law of Composition

From Schema 5.1 children can extract $a \times b = x$. In Example 1, for instance, the child recognizes the situation to be multiplicative, and therefore multiplies 4 $\times$ 15 or 15 $\times$ 4 to find the answer. This binary composition is correct if a and b are viewed as numbers. But, if they are viewed as magnitudes, it is not clear why 4 cakes $\times$ 15 cents yields cents and not cakes.

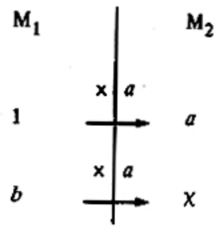
Unary Operation

It is most likely that children, especially young ones, do not extract a binary law of composition but rather a unary operation. This can be done in two different ways. Children can (*a*) use a scalar operator ($a \xrightarrow{\times b} x$) that consists of transposing in M_2, from a to x, the operator that links 1 to b in M_1.



SCHEMA 5.2

In Schema 5.2, $\times b$ is a scalar operator because it has no dimension, being a ratio of two magnitudes of the same kind; b cakes is b times as much as 1 cake, and the cost of b cakes is also b times as much as the cost of 1 cake. Or, (*b*) children can use a function operator ($b \xrightarrow{\times a} x$) that consists of transposing on the lower line, from b to x, the operator that links 1 to a on the upper line.



SCHEMA 5.3

In Schema 5.3, $\times a$ is a function operator because it represents the coefficient of the linear function from M_1 to M_2. Its dimension is the quotient of two other dimensions (e.g., cents per cake, kg per ha).[1]

FIRST-TYPE DIVISION

Schema 5.4 illustrates the first-type division, which is to find the unit value f(1).

[1] Another procedure for solving multiplication problems consists of adding $a + a + a \ldots$ (*b* times), but it is not a multiplicative procedure. It only shows that the scalar procedure relies upon iteration of addition. One does not find, in young children, the symmetric procedure $b + b + b \ldots$ (*a* times) because it is not meaningful.)

$$
\begin{array}{c|c}
M_1 & M_2 \\
1 & x = f(1 \\
a & b = f(a
\end{array}
$$

SCHEMA 5.4

Example 3. Connie wants to share her sweets with Jane and Susan. Her mother gave her 12 sweets. How many sweets will each receive?

$a = 3, \quad b = 12, \quad M_1 = \text{[numbers of children]}, \quad M_2 = \text{[numbers of sweets]}$

Example 4. Mrs Johnson bought some large peaches. Nine peaches weigh 2 kg. How much does one peach weigh, on the average?

$a = 9, \quad b = 2, \quad M_1 = \text{[numbers of peaches]}, \quad M_2 = \text{[weights]}$

This class of problems can be solved by applying a scalar operator $/b$ to the magnitude c (see Schema 5.5.).

$$
\begin{tikzcd}
M_1 & M_2 \\
1 & x \\
b \arrow[u, bend left, "/b"] & c \arrow[u, bend right, "/b"']
\end{tikzcd}
$$

SCHEMA 5.5

Some children, because mental inversion of the relationship $\times b$ into $/b$ is difficult, prefer to try to find x such that $x \times b = c$ (eventually by trial and error). This *missing factor procedure*, which is similar to *missing addend procedures* in subtraction problems, avoids the conceptual difficulty raised by inversion. But it is of value only for small whole numbers. Adults also use this missing factor procedure, when b and c are entries in the familiar multiplication table, for instance. But whereas they are able to shift to the canonical procedure c/b when necessary, young children usually fail.

Another procedure is available in the case of sharing objects: delivering them one by one to the participants or to different places in space. This can also be done mentally, even in other cases (by analogy), but it is inefficient and has no multiplicative character.

SECOND-TYPE DIVISION

Schema 5.6 illustrates second-type division, which is to find x knowing $f(x)$ and $f(1)$.

M_1	M_2
1	$a = f(1)$
x	$b = f(x)$

SCHEMA 5.6

Example 5. Peter has $15 to spend and he would like to buy miniature cars. They cost $3 each. How many cars can he buy?

$$a = 3, \qquad b = 15, \qquad M_1 = [\text{numbers of cars}], \qquad M_2 = [\text{costs}].$$

Example 6. Dad drives 55 miles per hour on the freeway. How long will it take him to get to his mother's house, which is 410 miles away?

$$a = 55, \qquad b = 410, \qquad M_1 = [\text{durations}], \qquad M_2 = [\text{distances}].$$

This class of problems can usually be solved by inverting the direct function operator and applying it to b, as shown in the Schema 5.7.

$$\begin{array}{ccc} M_1 & & M_2 \\ 1 & \xleftarrow{/a} & a \\ x & \xleftarrow{/a} & b \end{array}$$

SCHEMA 5.7

This procedure is difficult for children, not only because of the inversion prob-lem, but also because the inverse operator has a dimension (inverse of the direct one) that is unusual and harder to conceive (e.g., cars per dollar, hours per mile). Frequently, especially when the numbers are not small whole numbers, children prefer to find out how many times a goes into b, get the scalar operator, and transpose it in M_1. This avoids reasoning on inverse quotients of dimensions.

Children can also attempt additive procedures $a + a + a, \ldots$ until they get to b, then count the number of times they have added a.

RULE-OF-THREE PROBLEMS: GENERAL CASE

Schema 5.8 illustrates rule-of-three problems in the general case.

$$\begin{array}{c|c} M_1 & M_2 \\ a & b \\ c & x \end{array}$$

SCHEMA 5.8

Example 7. The consumption of my car is 7.5 liters of gas for 100 km. How much gas will I use for a vacation trip of 6580 km?

$$a = 100, \quad b = 7.5, \quad c = 6580, \quad M_1 = [\text{distances}], \quad M_2 = [\text{gas consumptions}].$$

Example 8. When she makes strawberry jam, my grandmother uses 3.5 kg of sugar for 5 kg of strawberries. How much sugar does she need for 8 kg of strawberries?

$$a = 5, \quad b = 3.5, \quad c = 8, \quad M_1 = [\text{strawberry weights}], \quad M_2 = [\text{sugar weights}].$$

This class of problems can be solved by different procedures, using different properties of the four-term relationship. They will be examined later in this chapter, under procedures for rule-of-three problems in the experiments.

It should already be clear that multiplication and division problems are simple cases of the more general rule-of-three class of problems in which four terms are involved, one of which is equal to one. In solving problems in this structure, students naturally use the isomorphic properties of the linear function:

$$f(x + x') = f(x) + f(x')$$
$$f(x - x') = f(x) - f(x')$$
$$f(\lambda x) = \lambda f(x)$$
$$f(\lambda x + \lambda' x') = \lambda f(x) + \lambda' f(x')$$

It is less natural for them to use the standard properties of the proportion coefficient:

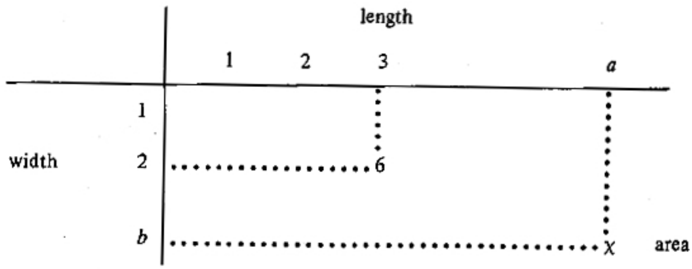
$$f(x) = ax$$
$$x = 1/a\, f(x)$$

Because the isomorphism properties appear to be more natural than the proportional coefficient properties, the expression *isomorphism of measures* is used to name and describe the simple direct proportion structure. This term enables us to distinguish very clearly this structure from the next ones, the *product of measures* and the *multiple proportion.* Whereas the product of measures structure and the multiple proportion structure involve three (or more) variables and a bilinear (or n-linear) function's model, the isomorphism of measures involves only two variables and is properly modelled by the linear function.

Product of Measures

The product of measures is a structure that consists of the Cartesian composition of two measure-spaces, M_1 and M_2, into a third, M_3. It describes a fair number of problems concerning area, volume, Cartesian product, work, and many other physical concepts.

Because there are (at least) three variables involved, this structure cannot be represented by a simple correspondence table like the one used for the isomorphism of measure structure. Rather it is represented by a double-correspondence table. For example, in the case of the area of a rectangle:



SCHEMA 5.9

Schema 5.9 reflects the double proportion of area to length and width indepen-dently. A similar relationship exists in the next structure (multiple proportion) but the choice and the expression of units do not obey the same rules. In the product of measures, there is a canonical way of choosing units. That is, in the case of finding the area of a square,

$$(1 \text{ unit of length}) \times (1 \text{ unit of length}) = (1 \text{ unit of area}).$$

The units of the product are expressed as products of elementary units; for example, square centimeters, cubic centimeters or, in Example 9:

$$f(1 \text{ boy} \times 1 \text{ girl}) = 1 \text{ couple}.$$

In the multiple proportion (to be described later), units do not generally have these properties.

Example 9. Four girls and 3 boys are at a dance. Each boy wants to dance with each girl, and each girl with each boy. How many different boy–girl couples are possible?

The different possible couples can be easily generated and classified by a double-entry table, and the proportion of the number of couples to the number of boys and the numbers of girls independently can also be made visible by a double correspondence table: the number of couples is proportional to the number of boys when the number of girls is held constant (parallel columns), and to the number of girls when the number of boys is held constant (parallel lines) (see Schema 5.10).

		Girls			
		L	M	N	O
Boys	A	AL	AM	AN	AO
	B	BL	BM	BN	BO
	C	CL	CN	CN	CO

Couples (Ex.9)

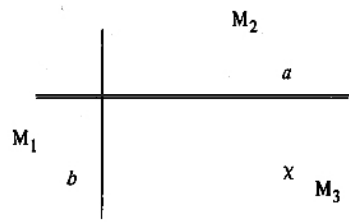
		Number of girls					
		1	2	3	4	5 …	n
Number of boys	1	1		3			
	2	2	4	6	8	10	$2n$
	3			9			
	m			$3m$			mn

Number of couples

SCHEMA 5.10

The Cartesian product is so nice that is has very often been used (in France anyway) to introduce multiplication in the second and third grades of elementary school. But many children fail to understand multiplication when it is introduced this way. The arithmetical structure of the Cartesian product, as a product of measures, is indeed very difficult and cannot really be mastered until it is ana-lyzed as a double proportion. Simple proportion should come first.

Two classes of problems can be identified, multiplication and division, the first of which is illustrated in Schema 5.11. Given the value of the elementary measures, find the value of the product-measure.



SCHEMA 5.11

Example 10. What is the area of a rectangular room that is 7 m long and 4.4 m wide?

$a = 7$, $b = 4.4$, $M_1 =$ [widths], $M_2 =$ [lengths], $M_3 =$ [areas].

Example 11. What is the volume of a pipe that is 120 cm long and has a cross-sectional area of 15 cm^2?

$a = 120$, $b = 15$, $M_1 =$ [section areas], $M_2 =$ [heights], $M_3 =$ [volumes].

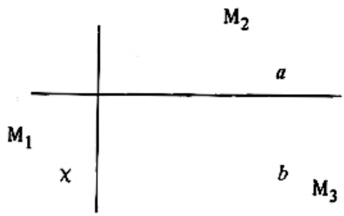
The solution $a \times b = x$ is not so easy to analyze in terms of scalar and function operators. It is a product of two measures both in the dimensional and the numerical aspects:

$$\text{area } (\text{m}^2) = \text{length } (\text{m}) \times \text{width } (\text{m})$$
$$\text{volume } (\text{cm}^3) = \text{height } (\text{cm}) \times \text{section area } (\text{cm}^2)$$

In the first structure (i.e., isomorphism of measures), we could not explain why multiplying cents by cakes would output cents and not cakes. This could only be explained either by the scalar operator (cents $\rightarrow$ cents) or by the function operator (cakes $\xrightarrow{\text{cents/cake}}$ cents).

In this second structure (i.e., product of measures), the landscape is different, and multiplying meters by meters outputs square meters; multiplying girl-dancers by boy-dancers outputs mixed couples of dancers.

The second class of problems, division, is illustrated by Schema 5.12. Given the value of the product measure and the value of one elementary measure, find the value of the other one.



SCHEMA 5.12

Example 12. The area of a pool is 150 m^2. Filling it up requires 320 m^3 water. What is the average height of water?

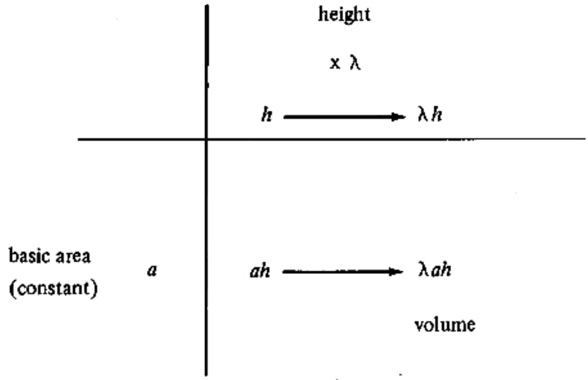
Here again, the division procedure cannot easily be described by a scalar or function operator. The dimension of the quantity to be found is the quotient of the dimension of the product by the dimension of the other "elementary measure."

$$\text{volume } (\text{m}^3) \; / \; \text{area } (\text{m}^2) = \text{height } (\text{m})$$

One way to explain the structure of the product is to see it as a double isomorphism or double proportion. Let us take the example of the volume of straight prisms.

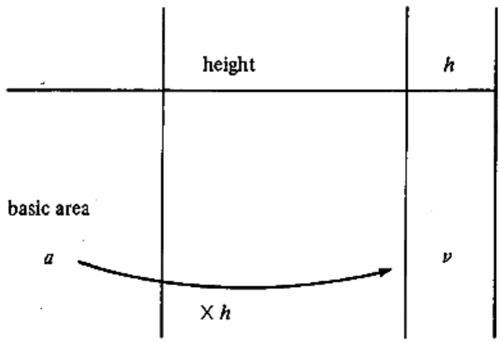
If height is multiplied by 2, 3 or λ, volume is multiplied by 2, 3 or λ (provided basic area ia held constant) (see Schema 5.13.)

Similarly, if basic area is multiplied by 2, 3 or λ', volume is multiplied by 2, 3 or λ' (provided height is held constant). If one adds basic areas of different prisms, volumes are also added (provided height is the same).



SCHEMA 5.13

These properties issue directly from the isomorphism properties of the linear function (scalar aspect, additive aspect). When height is held constant, ($\times\ h$) can be viewed as a function operator linking basic area to volume (see Schema 5.14.)



SCHEMA 5.14

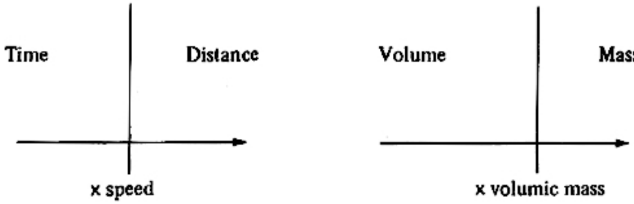
The same can be said for basic area when it is held constant. Although this analysis is a bit sophisticated, it shows that although product is different from isomorphism it can also be considered as a double isomorphism.

It follows that if height is multiplied by λ and basic area by λ', volume is multiplied by $\lambda\lambda'$.

Reciprocally, isomorphism can also be viewed as a product. For instance:

time × speed = distance
volume × volumic mass = mass.

These relationships are well illustrated by the function operator (see Schema 5.15.)



SCHEMA 5.15

Still, one should note that, in this case, speed and volumic mass are considered as constants and not as variables, whereas in the product (volume, for instance) both elementary measures (basic area and height in the case of volume) are variables. One must also keep in mind that, in the isomorphism structure, the quotient of dimensions is a derived magnitude and not an elementary one. If time × speed = distance, it is because speed = distance/time. If volume × volumic mass = mass, it is because volumic mass = mass/volume. However, in the product structure, at least in the product met by children at the primary and secondary levels (e.g., area, volume, Cartesian product), the elementary measures are really elementary and not quotients.

Multiple Proportion

The multiple proportion is a structure very similar to the product from the point of view of the arithmetic relationships: a measure-space M_3 is proportional to two different independent measure-spaces M_1 and M_2. For example:

1. The production of milk of a farm is (under certain conditions) proportional to the number of cows and to the number of days of the period considered.
2. The consumption of cereal in a scout camp is proportional to the number of persons and to the number of days.

Time is very often involved in such structures because it intervenes in many phenomena as a direct factor of proportionality (e.g., consumption, production, expense, outcome. But there are other factors; in physics, for instance:

$$P = kRI^2 \qquad \text{(Power, Resistance, Intensity)}$$

Whereas in physics multiple proportion phenomena can often be interpreted as products, this is not always possible in multiple proportion problems. Most of the time, no natural choice of units can provide $f(1,1) = 1$. For instance, there is no reason why a cow should produce 1 liter of milk per day, or a person eat 1 kg of cereal per day or per week. Usually there exists a coefficient k not equal to 1; $f(1,1) = k$.

In multiple proportion, the magnitudes involved have their own intrinsic meaning, and none of them can be reduced to a product of the others. There is no reason to interpret the double proportionality of the consumption of cereal to the number of persons and to the number of weeks as a dimensioned operation

$$\text{(i.e., cereal} = \text{persons} \times \text{weeks).}$$

Here again several classes of problems can be identified. I will only give examples, the analysis being similar to what has been explained earlier.

MULTIPLICATION

Example 13. A family of 4 persons wants to spend 13 days at a resort. The cost per person is $35 per day. What will be the expense?

FIRST-TYPE DIVISION

First-type division involves finding the unit value $f(1,1)$.

Example 14. A farmer tries to calculate the average production of milk of his cows during the 180 best days of the year. With 17 cows, he has produced 70,340 liters of milk during that period. What is the average production of milk per cow per day? (See Schema 5.16.)

		time	
		1	180
cows	1	x	
	17		70340 milk

SCHEMA 5.16

This division does not usually exist in the product of measures, because $f(1,1) = 1$, at least in the metric system.

SECOND-TYPE DIVISION

Second-type division involves finding x knowing $f(x,a) = b$ and $f(1,1)$.

Example 15. A scout camp has just received 500 kg of cereal. The allowed distribution of cereal is 0.6 kg per person per week. There are 236 persons in the camp. How long will the cereal last? (See Schema 5.17.)

persons \ time	1	x
1	0.6	
$a = 236$		$b = 500$ cereal

SCHEMA 5.17

The bilinear function is an adequate model for both the product of measures and the multiple proportion. One hypothesis is that it implies more complex operations of thinking than does the linear function. Another hypothesis is that the product of measures raises its own difficulties that are not reducible to those of the multiple proportion.

In the next part of this chapter, I report experiments on problems that can be analyzed by these structures. Sometimes problems are combinations of different structures. The above analysis is a first approach to these problems. A comple-mentary analysis will be made later. The value of the magnitudes involved, the concept of average, and the reference context are also important characteristics of problems.

Experiments

This section describes several experiments, conducted during the past 4 or 5 years, showing either results on the comparative complexity of problems and procedures, or the evolution in the classroom of conceptualizations and procedures in a dialectical relationship with situations.

Isomorphism, Product, and Multiple Proportion

The very first experiment performed by our research group (Vergnaud, Ricco, Rouchier, Marthe, & Metregiste, 1978) aimed at comparing the difficulty of different problems and evaluating the stability of procedures used by students. It consisted of different versions of the same three problem-structures:

1. Volume: calculation of the volume of a right parallelipiped: $x = a \times b \times c$.
2. Direct proportion: among three measure-spaces; calculation of x knowing $f \circ g(x) = c$ $\quad x = c/(a \times b)$ (See Schema 5.18.)

M_1	M_2	M_3
	1	a
1	b	
x		c

$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$

SCHEMA 5.18

3. Double proportion: calculation of a magnitude proportional to time and to another magnitude. (See Schema 5.19.)

		time	
		1	c
	1	a	
other magnitude	b		x

SCHEMA 5.19

The first version of these structures was a complex problem (with several questions), chosen from a handbook for 11–12-year-olds, involving all three structures.

> "Central heating is being installed in a house; the dimensions are length = 18 m, width = 6 m, height = 4 m.
>
> 1. A radiator is made of 8 elements. Each element can heat 6 m^3. How many radiators are needed?
> 2. The average consumption is 4 kg coal per day for each radiator. The heating period runs from October 1 to April 15. How much coal will be used?"

The other versions were single questions, built ad hoc for the sake of comparison:

Volume: "What is the volume of water used to fill up a rectangular pool: Length = 17 m, width = 8 m, depth = 3 m."

Direct proportion: "A luxury train should contain 432 first-class seats. Each car has 8 compartments and each compartment has 6 seats. How many cars are needed?" Another version was also used.

Double proportion: "A farmer owns 5 cows. They each yield an average of 23 liters of milk per day during the 180 best days of the year. How much milk does the farmer produce during this period?"

Among 84 students (11–12 years old) we could find the following hierarchy: The easiest problems were the direct proportion and the double proportion ones, although the direct proportion problem needed either two divisions, or a multiplication and a division. Both problem types were solved by $\frac{2}{3}$ of the students in the simple version and $\frac{1}{3}$ in the complex version.

The most difficult problem had to do with volume, both in the simple and the complex versions (success rate of $\frac{28}{84}$ and $\frac{28}{84}$), although the calculations were the simplest.

We analyzed the different procedures used and categorized them, so as to compare the stability of these categories on two (or three) versions of the same problem-structure. Many students had a *perimetric* representation of volume, adding lengths together and trying to take into account as many sides as possible (either by multiplying 2, 3 or 4 times the sum $L + W + H$, or by adding 1, 2 or 4 times the height to the perimeter of the base, or by any other combination). Some students also had a *surface* representation (adding areas) or a *mixed* representation (multiplying perimeter by height, for instance). We found only 50% stability on classes of procedures, which is far beyond random coincidence; but still rather weak.

For the direct proportion we also found different classes of procedures. Most errors involved partially correct procedures that were not carried out to the solution. The stability was quite good for one of the correct procedures, which had the easiest physical meaning, and comparatively weak for the others.

As for the double proportion, we found better performance in relation to the time factor. Both consumption of coal and production of milk are conceived as proportional to time, even though the other factor involved (number of radiators, number of cows) did not appear to be difficult.

Another finding of this experiment was that most procedures used by students, even when they were wrong, had a physical meaning. We very rarely found meaningless calculations.

A Variety of Procedures for Rule-of-Three Problems

The experiment (Vergnaud, Rouchier, Ricco, Marthe, Metregiste, & Giacobbe, 1979; Vergnaud, 1980) was designed to test the hypothesis of a better availability of scalar procedures compared with function procedures. It also permitted us to make an extensive description of procedures (correct or incorrect) used by students. The problem was the following:

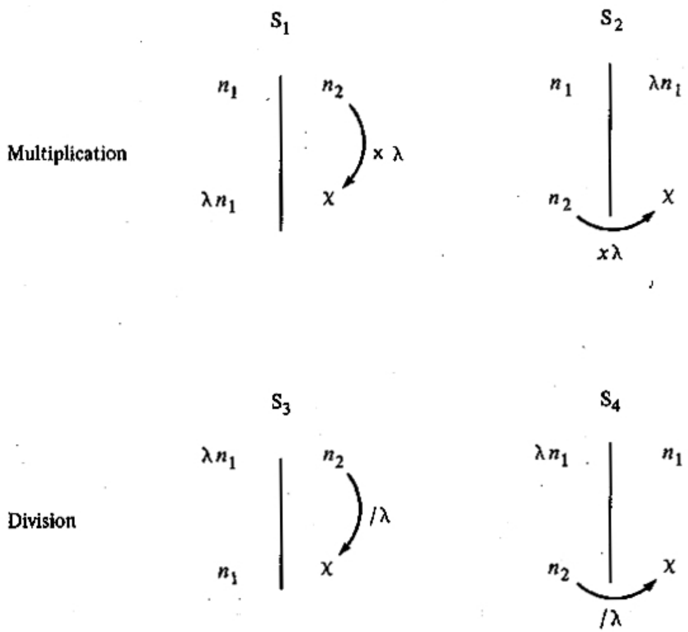
"In a hours the central heating consumption is b liters of oil. What is the consumption in c hours?" (See Schema 5.20.)

M_1	M_2
time	consump
a	b
c	x

SCHEMA 5.20

By choosing adequate values for a, b and c, it is possible to simplify either the scalar ratio ($\frac{c}{a}$) or the function ratio ($\frac{b}{a}$), or both, or none.

In order to test the hypothesis that scalar procedures would be more easily and frequently used than function procedures, we used problems with a simple scalar ratio (3 or 4) and a complex function ratio (12, 13) and problems with a simple function ratio and a complex scalar ratio. This was done for multiplication and division, resulting in four cases in all. (See Schema 5.21.)



SCHEMA 5.21

We used a total of 16 problems. Examples are as follows. (See Schema 5.22.)

$$\begin{array}{c|c} 9 & 108 \\ 36 & \end{array} \qquad \begin{array}{c|c} 7 & 21 \\ 84 & \end{array} \qquad \begin{array}{c|c} 18 & 78 \\ 6 & \end{array} \qquad \begin{array}{c|c} 32 & 8 \\ 104 & \end{array}$$

$S_1 \qquad S_2 \qquad S_3 \qquad S_4$

SCHEMA 5.22

We counterbalanced the order of the four problems that each student had to solve. Four groups of 25 students each (one group for each secondary compre- hensive school grade, from 11–12-year-olds to 14–15-year-olds) participated in the experiment.

According to our hypothesis, we expected S_1 to be easier than S_2, and S_3 to be easier than S_4. We also expected to find differences in the use of the different possible procedures. We also meant to describe the evolution of success rates and procedures along the four grades of early secondary school.

Table 5.1 shows very clearly that there is no difference between the S_1 and S_2 problems. This result contradicts our hypothesis that an easy scalar ratio would enable children to solve S_1 problems more easily than S_2 problems (easy function ratio).

The situation is different for S_3 and S_4. Whereas S_3 problems are mastered nearly as well as S_1 and S_2, there is a big drop in the success rate for S_4. Unfortunately, this drop may be due to two different factors: the difficult scalar division on one hand, and the fact that S_4 is the only situation where $f(x) < x$ (as can be seen in the numerical examples above).

One interesting thing is the regular, slow evolution from younger students to older ones. This shows that a psychogenetic approach is useful in studying the acquisition of mathematical skills at the secondary school level. Even when these

TABLE 5.1

Success Rates for Rule-of-Three Problems (%)

	Problems			
	S_1	S_2	S_3	S_4
Grades	$\begin{array}{cc} n_1 & n_2 \\ \lambda n_1 & \end{array}$	$\begin{array}{cc} n_1 & \lambda n_1 \\ n_2 & \end{array}$	$\begin{array}{cc} \lambda n_1 & n_2 \\ n_1 & \end{array}$	$\begin{array}{cc} \lambda n_1 & n_1 \\ n_2 & \end{array}$
6th graders (11–12 years old)	39	39	29	16
7th graders (12–13 years old)	64	55	59	36
8th graders (13–14 years old)	65	69	69	35
9th graders (14–15 years old)	82	85	74	56
	63	63	58	36

skills are taught, the development is slow, and it takes students a few years to deal successfully with the different numerical cases. Some students still fail to handle the simplest situations even at the end of early secondary school, but most of them progress regularly. The most difficult case used here, which is not the most difficult that one can meet, even with whole numbers, is mastered by a majority of students only at the last early secondary school level (14–15-year-olds).

Procedures

More interesting is the variety of procedures we observed (over 25 kinds used by students). We classified correct procedures into five subcategories; we tried to classify incorrect ones into meaningful subcategories, but this was not always possible.

In the following description, we use letters a, b, c and x: a and c are time-measures, b and x are consumption measures as shown in Schema 5.23.

M_1	M_2
time	consumption
a	b
c	x

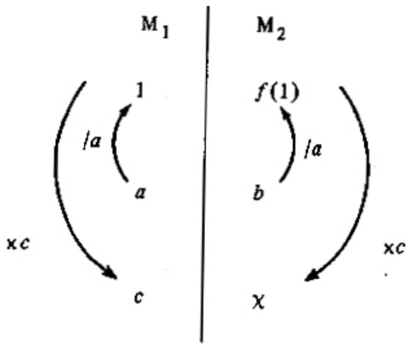
SCHEMA 5.23

Correct Procedures

S *Scalar:* The student calculates $(c/a = \lambda)$ (in S_1 and S_2) or $(a/c = \lambda)$ (in S_3). This calculation can be made explicitly either by dividing or by using the missing factor procedure (see Schema 5.23). It can also be performed mentally. Afterwards, the student calculates $x = \lambda \times b$ or $x = b \times \lambda$ (in S_1 and S_2) or b/λ (in S_3).

F *Function:* The student calculates $(b/a = \lambda)$ (in S_1 and S_2) or $(a/b = \lambda)$ (in S_4), either explicitly or mentally and then $x = \lambda \times c$ (S_1 and S_2) or c/λ (S_4).

U *Unit value:* The student performs the same calculations as in F, but he explains that (b/a) is the unit value $f(1)$. This procedure is scalar in character, although the calculations are the same as in F. (See Schema 5.24.)



SCHEMA 5.24

R *Rule of three:* The student calculates $(b \times c)/a$ or $(c \times b)/a$ (multiplication first). This well-known algorithm is rarely used. We explain why later.

SD *Scalar decomposition:* The student tries to decompose magnitude c as a linear combination of different other magnitudes: multiples of a or fractions of a.

Example: $a = 32$, $b = 8$, $c = 104$ (boy; 14 years old).

Protocol	Comment
$32 \times 3 = 96 + 8 = 104$	$104 = (3 \times 32) + (\frac{1}{4} \times 32)$
	$\Downarrow$
$86 = 24$ liters $+ 8/4$	$x = (3 \times 8) + (\frac{1}{4} \times 8)$
$= 26$ liters	$= 24 + 2$

Although the equations in the protocol are all wrong, the procedure is efficient and shows the use of a powerful theorem (see comment):

$$(f(\lambda a + \lambda' a) = \lambda f(a) + \lambda'' f(a)$$

This procedure is very often used by students (even at the primary school level) when they cannot think of the function-operator. The decomposition can also be multiplicative:

$$f(\lambda\lambda' a) = \lambda\lambda' f(a)$$

The properties of numbers are of course very important in the emergence of such procedures but it is also important to notice that these procedures cannot be explained by pure numerical properties. Numbers are magnitudes. As a matter of fact, if the numbers did not represent magnitudes of qualitatively different types of quantities, then one should also find function decomposition procedures ($b = \lambda a + \lambda' a$). This is never the case; b cannot be conceived as a linear combination of magnitudes of a different kind.

Incorrect Procedures

Many incorrect procedures are based on some aspects of the indicated real situation. We thought it might be interesting to classify these incorrect procedures in order to see what sort of features were the most salient.

S *Erroneous scalar:* The student uses a scalar ratio or difference, either (c/a) or $c - a$ or (a/c) or $a - c$, and either gives it as an answer, or multiplies it by b, or adds it to b, or divides b by it, or subtracts it from b.

Example: $a = 7$, $b = 21$, $c = 84$ (girl; 16 years old).

$$84h - 7h = 77\ h$$

The consumption is 77×21 liters $= 1617$ liters.
We also considered that multiplying b by an arbitrary number, approximately equal to c/a, could be classified in this category.

F *Erroneous function:* The student uses a function ratio or difference, either (b/a), or $b - a$, or (a/b), or $a - b$, and either gives it as an answer or applies it to c.

S′F′ *Erroneous scalar and function:* The student makes a calculation $b \times c$, forgetting or cancelling division by a or makes a combination of erroneous scalar and function operations.

Example: $a = 8$, $b = 32$, $c = 104$ (girl; 13 years old). (See Schema 5.25.)

$$8\overline{)32}^{\,4} \qquad\qquad \begin{array}{r} 13 \\ 8\overline{)104} \\ \underline{8} \\ 24 \\ \underline{24} \end{array} \qquad\qquad \begin{array}{r} 13 \\ \underline{\times 4} \\ 52 \end{array}$$

SCHEMA 5.25

In $104h.$, the consumption is 52 litres.

I *Inverse:* The student uses the inverse ratio a/c instead of c/a, or c/a instead of a/c, or b/a instead of a/b, etc.

Example: $a = 21$, $b = 7$, $c = 90$ (girl; 14 years old).

$$21{:}7 = 3$$

$90 \times 3 = 270$ liters.

P *Erroneous product:* The student multiplies c and a, or b and a, which has no physical meaning at all.

Q *Erroneous quotient:* The student divides c by b, or b by c, which again has no meaning.

O *Others:* Procedures that could not be classified elsewhere.

Looking at these incorrect procedures, one can see some differences. Erroneous scalar and function, and inverse procedures are less "silly" than erroneous products and quotients. One can make better sense of them. Comparing two magnitudes of the same kind by looking at the difference $c - a$ instead of the ratio c/a also makes better sense than looking at $b - a$ instead of b/a.

Table 5.2 shows the distribution of procedures on each problem-structure for all grades together. The most striking fact is that scalar procedures are more frequently used than are function procedures, even for problems in which the function-operator is very simple (S_2 and S_4). This is true for procedure S alone, but it is still more striking if one considers S, V, SD and S′ together, compared with F, FD and F′.

This fact has been found or observed by other authors (Freudenthal, 1978; Lybeck, 1978) and discussed by others (see Karplus, chapter 3, this volume. Noelting, 1980 a,b) under the distinction between *Within* ratios and *Between* ratios. The discussion may be confused if one mixes up problems of comparison and problems of calculation, in which the answer is a certain magnitude. I return to this point when considering problems of fractions and ratios. For the time being, I just stress the convergence of Lybeck's (1978) results with ours, and the fact that the analysis in terms of *isomorphism of measures* is the most powerful and the most general one.

TABLE 5.2

Distribution of Procedures for Rule-of Three Problems (%)

Procedures	Problems			
	S_1	S_2	S_3	S_4
Correct				
S Scalar	41	32	38	16
F Function	11	14	6	8
U Unit value	9	14	10	5
R Rule of three	1	1	2	1
SD Scalar decomposition	1	0	0	4
FD Function decomposition	0	0	0	0
Incorrect				
S′ Erroneous scalar	8	6	20	5
F′ Erroneous function	0	1	4	10
S′F′ Erroneous scalar and function	10	9	0	5
I Inverse	2	2	1	11
P Erroneous product	3	4	1	3
Q Erroneous quotient	3	4	2	1
O Others	11	13	16	31
	100	100	100	100

Another striking fact is that only 1% of the students use the rule-of-three algorithm. This also has been found by others (Hart, 1981), but it must be explained. My view is that it is not natural for students to multiply b by c and then divide by a if one considers b, c and a as magnitudes. It is fair to do this in the set of numbers because of the equivalence of different calculations $(b \times c)/a$, $c/a \times b$, $b/a \times c$. But children do not think of b, c and a as pure numbers. They see them as magnitudes and there is no meaning for them in multiplying b liters of oil by c hours, whereas they can more easily figure out scalar ratios c/a or even function ratios b/a.

We also found that the older students, although they had studied the linear function $f(x) = ax$ and the proportion coefficient, used the scalar procedure (in its different versions S, V, and SD) more often than did the younger students. For details on the first ideas of children on linear functions, see Ricco, 1978.

Volume: A Difficult Concept

This experiment, which was completed recently and will be published with more details at a later date, consisted of 80 individual interviews with secondary school students: 10 boys and 10 girls in each of the four grades, from 11–12 to 14–15-year-olds. Its aim was to obtain a varied and meaningful picture of students' skills and representations so as to be in a better position to make a series of didactic situations on volume for seventh graders (12–13-year-olds). Most of the questions concern trilinear properties of volume but we also tested the availability of the formula for parallelepipeds, and tried to obtain definitions of volume.

Because we intended to explore aspects varying in complexity, we used a branching program of items, posing more difficult questions to successful students and easier questions to those who had failed.

THE INTERVIEW

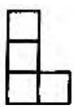
The interview started with the estimation of the volume of an aquarium, placed on the table. The student could use a meter stick to measure the dimensions of the aquarium, which were not given. We recorded what dimensions he (or she) did actually measure, and the sort of calculations he (or she) performed. The student was then asked to estimate the volume of the classroom (rectangular); no dimensions were given. We were not so much interested in the estimation of length, width, and height as in the calculations. The students were asked to explain how they arrived at their answers and were also asked, "What is volume for you?" This could also be repeated as, "If you had to explain to a younger fellow, what would you tell him?"

Students who failed to find a correct multiplicative calculation for the aquarium and the classroom were then asked a simpler question with blocks. A large quantity of blocks was displayed on the table, and then was hidden away in the experimenter's bag. The student was asked: "How many blocks should I give you, for you to make a straight box, as full as a box of sugar pieces, 3 blocks wide, 4 blocks long and 2 blocks high?" This item was obviously aimed at helping students find the correct calculation by using the blocks as a model (paving space with unit blocks).

Students who did not succeed on these items did not continue. The others were then asked the following problem: "Mr. Dupont has a small aquarium in his kitchen, and a large one in his den. The den one is twice as long, three times as wide and twice as deep as the kitchen one. How many times is the den one larger than the kitchen one?" We recorded the answers and the explanations.

There were two more items on the trilinear aspects of volume:

1. Two spheres ($D = 4$ cm and $d = 2$ cm) were shown, and the first question was: "How many times is the volume of the big one larger than the volume of the little one?" The second question concerned the weight. The student was then given several little spheres and a plastic toy scale. The big sphere was placed on one plate: "How many little ones should you put on this side to get the equilibrium?" (All spheres were solid and made of the same wood.)



SCHEMA 5.26

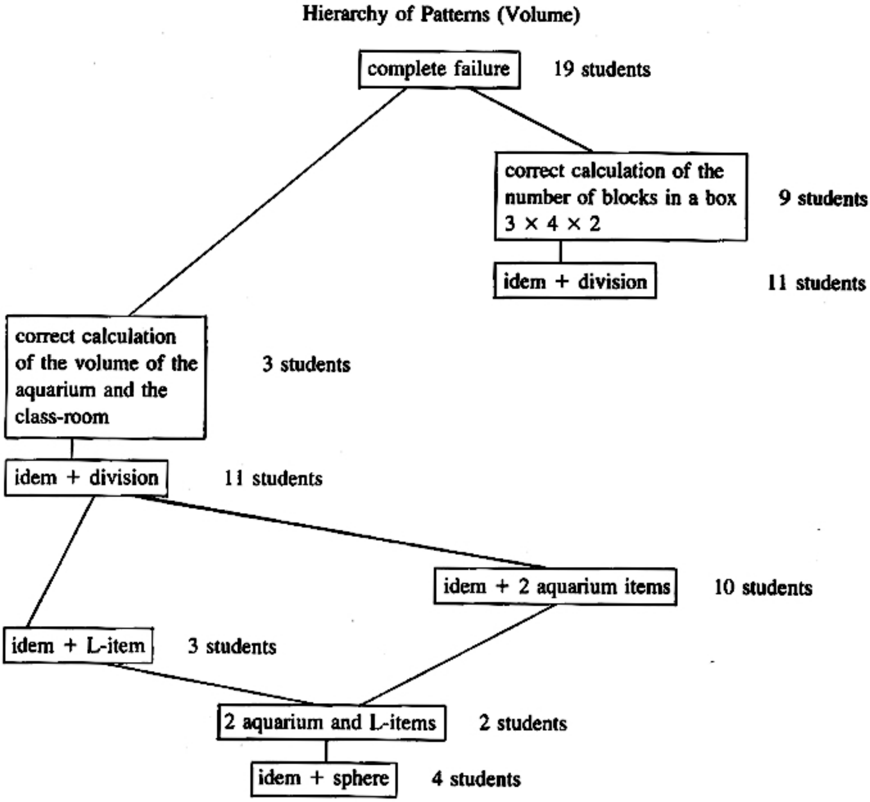
2. An L-shape, made of 4 blocks (see Schema 5.26), and an enlarged version of the same shape (twice as long, twice as wide, twice as thick) were shown. The little L was permanently on the table; the big one was shown and immediately hidden. The question was: "How many blocks are there in the big L?" The difficulty of this item was expected to be intermediary between the two-aquarium item and the sphere item.

Finally a question was asked on division: "The volume of a box is 60 blocks. It is 3 blocks wide and 4 blocks high. How long is it?"

RESULTS

The response patterns are compatible with the hierarchy of items summarized in Schema 5.27.

Notice that 19 students failed completely and that only a small minority were able to handle trilinear aspects of volume. Most students just used the formula for the calculation of the volume of parallelepipeds, or were reminded of it through the block model. Only 4 students were successful in the sphere item, and the L-

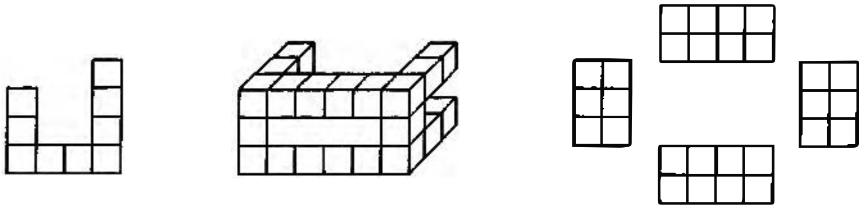


SCHEMA 5.27

item was also very difficult. There was no difference between the two first items; it was not easier to calculate the volume of the room in cubic meters with small numbers ($6 \times 4 \times 3$) than the volume of the aquarium in cubic centimeters with larger numbers ($40 \times 17 \times 20$).

Most 11–12-year-olds failed completely, but there were also total failures among older students. The jump is important from 11–12 to 12–13-year-olds (volume is taught again to 12–13), but the two-aquarium item was very difficult until ninth grade. Although volume is not taught any longer after the seventh grade (12–13) and is supposed to be well known by students (which is not the case), some skills go on improving, slowly.

We classified answers and procedures used by students and found again, in the direct calculation of volume, the perimetric representation, the area representa-tion, and the mixed representation that we had already observed in the first experiment. This was even the case with the block item; many young students made drawings in which perimetric and area models appeared clearly (Schema 5.28).



SCHEMA 5.28

The perimetric representation disappeared completely in the last grade, but this was not the case with the area representation: confusions between area and volume seem to be long lasting.

The stability of the procedures used for the calculation of the volume of the aquarium and of the classroom was quite good (75%). An expected finding was that some students could not express their results in cubic meters or centimeters, but would express them as meters or centimeters, or even centiliters.

The definitions of volume were varied. It is difficult to translate into English the exact wording of students. Table 5.3 is an attempt at such a translation. It shows a very interesting variety of definitions and also a good evolution from perimetric to volumic representations.

The two-aquarium item produced interesting results: out of 39 students who responded to the item, one finds 17 correct responses: 10 students gave the answer, $2 \times 3 \times 2 = 12$; 4 students paved the big aquarium mentally with the little one; and 3 students attributed hypothetic dimensions to the little one, calculated the dimensions of the big one, the volume of both, and then the ratio. This last "detour" shows how conceptually difficult it is to compose ratios in this structure.

Of the 22 incorrect responses, 9 students gave the additive answer, $2 + 3 + 2 = 7$; 3 students gave the "average" answer, "between 2 and 3;" and 1 student gave the modal answer, 2. Other students either tried a good procedure but got mixed up, or repeated the information given by the experimenter, or decided that it was impossible to say, because "one did not know the measure of the little aquarium." This very last answer confirms the conceptual difficulty of composing ratios in the absence of any information on the associated magnitudes.

As we expected, the sphere item was very difficult. One interesting fact was that students who gave the correct answer, after having hesitated for a while, used the gestural metaphor of putting together little spheres to fit the space occupied by the big sphere. This is of course a pure metaphor, borrowed from the cube model, but it does help. The most frequent incorrect answers were 2 and 4, as expected. The answers for the weight were not any different from those given for the volume; the students were deeply surprised when they discovered that 8 little spheres were necessary to make the equilibrium. None of the 19 subjects

TABLE 5.3

Definitions of Volume

	Age				
	11–12	12–13	13–14	14–15	Total
The room occupied by			1	1	2
The space occupied by			2	2	4
Length by width by height			1		1
The three dimensions of which one can move			1	1	2
The inside of		1	3		4
The interior of	1	2	2	2	7
The quantity contained in (a bottle, a room, a box . . .)	1	4	1	2	8
The capacity of	1	2	2	2	7
What is contained		2		1	3
A quantity (pencils, sheets, air, water . . .)	3	3		1	7
Filling up something (with water, cubes . . .)	1	2			3
The weight	1		1		2
The mass in the air		1	1	2	4
The area	1			1	2
All the room (gestures indicating surface)			2		2
The surface	2				2
The total of square meters	1				1
All the length of the room	1				1
The whole outline, contourline	2	1		1	4
What the room measures altogether (gestures)	2				2
All the sides	1		1		2
All the dimensions		1	1		2
Others	1		1		2
No answer	1	1	—	3	5
Total	20	20	20	20	80

that had given a wrong answer was able to explain correctly this astonishing fact: they used ad hoc explanations (for instance: multiplication of D by d) or refused to make any comment.

For the L-item, most students tried to imagine mentally the Big L and to count the blocks; most of them failed. Only four students attempted to compose directly the similarity ratio; one of them failed.

In conclusion, most students do not master the simplest elementary trilinear properties of volume, in the case of the parallelpiped, until the ninth grade (14–15-year-olds). Only a few of them can understand more difficult cases (L-item and sphere). Even the direct calculation of volume or the inverse calculation of one dimension knowing the volume and the other dimensions are still difficult for 13- to 15-year-olds, although they have been taught it. The multiplicative three-dimensional model contradicts more "natural" models such as the perimetric model and the area model. It is important to help students overcome these models and differentiate them from one another and from volume ideas. The difficulty of the concept of volume is considerably underestimated by teachers and programs, certainly in France. Volume is a good example of what was said in the introduction to this paper about how concepts develop over a long period of time.

Didactic Experiments

In the short space available in this chapter, I can only illustrate with a few examples the kind of experiments that we have developed inside the classroom. We usually have two aims: (*a*) to improve students' knowledge and know-how; (*b*) to make reliable observations and discover didactic facts, that is, facts concerning transmission and acquisition of knowledge.

The first important part of our work consists of arranging a series of didactic situations and making as explicit as possible both our didactic objectives and our hypotheses about what might happen. All members of the research team (mathematics teachers, psychologists, and mathematicians) participate in the choice of conceptions and ideas, and in the choice of situations (context, values of the different situation variables, order of problems, allowed suggestions, and so on). For instance, in the didactic sequence on volume, described below, we explicitly meant to start from a unidimensional conception of volume, as a quantity that can be compared and measured directly, to arrive at a tridimensional conception of the volume of parallelepipeds and prisms (see Rogalski, 1979; 1981, pp. 120–125, for interesting results). We also considered two important intermediary steps: the paving of the parallelepiped (as a natural transition from the unidimensional to the tridimensional conception) and the differentiation between volume, lateral area, and edge–periphery of a building (see the architect's problem). We also organized deliberately different difficulties and jumps in the questions posed to students.

But, before describing any situations or results, I need to clarify a few methodological points.

All situations are carefully described and written down for the teachers and the observers, with the different phases of each 50-minute lesson, the formulation of questions, and suggestions authorized to help students. This is the sort of care taken by psychologists when they interview subjects; although it is impossible to copy that model, we find it necessary to get as close to it as possible.

Students are usually divided into groups of four. Most of the lesson time is devoted to small group work. At some previously determined moments, explanations to the whole class are delivered, at the blackboard, by students representing their group, or by the teacher. During these phases, the different group conclusions are summarized and new questions raised.

When a series of 50-minute lessons has been programmed, it is run (with minor changes) in different classes to permit comparisons and the discovery of recurrent facts. By *different classes* we mean different classes of the same grade, and eventually classes of different grades. In each class, one group is permanently video recorded whereas (most) other groups are watched by one "observer" each. The observer tries not to intervene, but still does help the group when necessary. After each lesson, observations are discussed. The whole-class work phases are also video recorded.

One of the aims of such complex experiments is to observe regularities and establish and interpret reliable didactic facts. Pretest–posttest evaluations can also be organized, but this is not essential to the methodology.

AN EXAMPLE: A DIDACTIC SERIES ON VOLUME FOR SEVENTH GRADERS

Volume is a geometric–physical magnitude that can be, in certain cases, directly compared and measured. For instance, bottles, glasses, cups, vases, and other kinds of containers can be compared. This is not a difficult job for children. More difficult is the comparison of *full volumes*, such as stones, pieces of plasticine, or complex block shapes like those used in this experiment, and the comparison of full volumes with *hollow volumes*, or containers. These comparisons must usually be achieved with the help of some liquid (water, for example), and involve indirect comparisons.

In the first lesson, each group had first to compare four containers and to order them, $A < B < C < D$. Because these containers had been cut from tops and bottoms of different plastic bottles, the order was not obviously perceptible and students had to fill them with water to decide. During this phase we observed that although the task was easily done by seventh graders, the transitivity axiom was not always used and some redundant comparisons were made.

Each group was then given two full volumes, a block shape, and a plasticine one and had to place them in an ordered series:

Either E and H such that $E < A$, $C < H < D$
or F and G such that $A < F < B < G < C$.

The group sometimes had difficulties comparing full volumes with each other, and had even greater difficulties in comparing them with hollow volumes. The reason for these difficulties is the fact that such comparisons require complementary volumes and reasoning on complements. If F is put into X, and then G into X (X being a large container filled with water), then from $X - G < X - F$ the correct conclusion is $F < G$, not $F > G$.

Still more difficult is the comparison of a hollow with a full volume, B and F for instance, because one may have to find a liquid equivalent of F by complementing $X - F$ to X (double complement), or by reasoning on levels of water in X and planning a sequence of actions and measurements that is sometimes too complicated for the group, especially if two students have different plans. The protocols cannot be described in detail here, but we were struck by the unexpected difficulty of the task. Some children did not even bother to keep F totally under water, and drew inadequate conclusions because of this faulty procedure. In one class, no group was able to give a correct ordered series of full and hollow volumes. In other classes it was difficult for most groups. The task was easily done in only one class. Clearly for seventh graders (and probably for older students), the results show the relevance of this kind of situation and type of task.

In the second lesson, children were asked to measure all volumes. There are several ways of associating a number with an object. The volume formulas are one way to do it. In this lesson, a more direct method was used: two different capacity units were used for containers, different groups using different units. A natural solid unit (the block unit) was used for the block shapes E and F. It is fairly easy to count the number of capacity units in a container or to count the number of blocks in a block shape. More difficult is the association of a number of capacity units to full volumes or a number of blocks to containers and to plasticine shapes. These tasks involve either reasoning on complements or solving rule-of-three problems. So there are three different units: two liquid units, u_1 and u_2, and the block unit, u_3, related as follows:

$$2u_1 = 3u_2$$
$$u_1 = 18u_3$$

These equivalencies were not given and had to be discovered by students during the phase of calculation of the value of each volume in each unit system. The only way to complete this task is to find a common reference point: same volume measured with different units. This is not difficult for the first equivalency and for containers A, B, C, and D because the containers are the same in all groups and can be easily measured with u_1 and u_2. It is much more difficult for full volumes and for u_3 measures.

Table 5.4 summarizes the situation; it is a table of proportional numbers in which students have to fill empty squares. This table is not given to the students; it is drawn at the end of the second lesson, or at the beginning of the third lesson to summarize the results. Some scalar and function procedures used by students to calculate u_1 and u_2-measures of A, B, C and D are also described: (a) scalar procedure: Suppose you know that $m_1(A) = 6$, $m_2(A) = 9$ and $m_1(D) = 12$; because D is twice as big as A, then $m_2(D) = 18$; (b) function procedure: because $m_2(A)$ is $1\frac{1}{2}$ times as much as $m_1(A)$, then $m_2(D) = 12 \times (1\frac{1}{2}) = 18$. The above procedures can also be expressed in terms of column-to-column operators or line-to-line operators in the table.

The aim of the next two lessons was to help students move from a unidimen-

TABLE 5.4

Measures of Recipients A, B, C, and D and Full Volumes E and F

	E	A	F	B	C	D
u_1	—	6	—	8	10	12
u_2	—	9	—	12	15	18
u_3	72	—	126	—	—	—

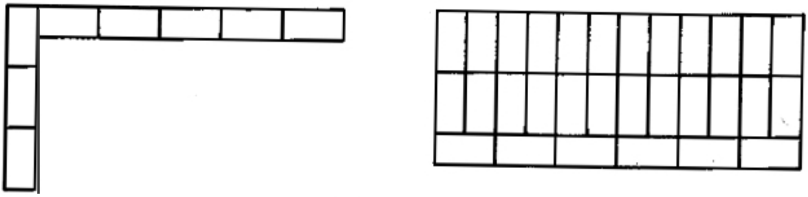
sional conception of volume to a tridimensional one, to coordinate both conceptions, to compare different ways of paving and different sorts of elementary units, and to analyze the meaning and the homogeneity of the formula.

Two parallelepipedic boxes, A ($180 \times 120 \times 60$) and B ($180 \times 90 \times 75$), were used. Large numbers of two sorts of elementary units were provided: small parallelepipeds ($30 \times 20 \times 15$) and small cylinders ($d = 15$; $h = 30$). Measures were expressed in millimeters. They were not given to students.

Students were asked to measure and compare A and B. The idea is that both kinds of elementary units may be used, but that cylinders do not occupy all empty space and that Box B cannot be regularly and completely paved in all directions whereas Box A can. Further, the complete paving of A and B takes a long time, which can be saved by counting the number of elements that can be put in the length, in the width, and in the height, then multiplying. Finally, cubic units are indifferent to the orientations and optimize the procedure. This last idea is then tested with blocks of 1 cc. Although a large number of such cubic blocks was provided to each group, this number is not large enough for the complete paving of Boxes A and B, and the use of the formula is then unavoidable. The complete paving would take too much time anyway.

In a later phase, questions were raised concerning enlargement of parallelepipedic volumes by a ratio of 2, 3, or 10 on lengths—2^3, 3^3, and 10^3 on volumes)—and concerning all different possible combinations to make a parallelepiped of 24 blocks, or 48 blocks.

We observed most interesting behaviors during the paving process. In starting to work, many groups did not even think that it would save time not to pave completely Box A and Box B (they probably enjoyed paving the boxes). But unexpectedly, we also found groups that did not even find it necessary to make their paving regular; they would pave one line in length, then one line in width, along the same dimension of elements (see Schema 5.29 for examples of noncanonic pavings); or they would try to fill the box as completely as possible by paving the box with several lines regularly displayed and one more line differently organized. This of course does not allow them to make any simple multiplicative calculation.



SCHEMA 5.29

Another interesting cognitive conflict occurred when the students tried to count the number of parallelepipeds in the length, in the width, and in the height. Beside the fact that they might arrange the elements along the length in all three dimensions, as mentioned above, some students were faced with the contradiction between the additive direct conception of volume and the multiplicative tridimensional conception. This contradiction was made especially obvious by a block-in-the-corner incident that took place in many groups. It usually appeared as a conflict between two members of the group, one of them saying that the block-in-the-corner should not be counted twice: "as it has been counted in the length, one should not count it in the width, or in the height," the other member of the group saying (without always being able to explain why) that it should be counted in the length, in the width, and in the height. The first position results from an additive direct conception of volume (in which it is actually true that the same partial volume cannot be counted twice), and the second position is associ-ated with a multiplicative conception, in which blocks along the length, width, or height are not volumes but instruments to measure length, width, and height. The first conception is also reinforced by a perimetric view of volume as a composi-tion of edges. This conflict, observed many times in different groups and differ-ent classes, is typically a didactic fact, due to the situation in which both uni-dimensional and tridimensional conceptions can be used. We would not have expected it to arise naturally in groups of 12–13-year-olds.

We also observed interesting behaviors mentioned before in the experiment on volume, concerning enlargement by a similarity ratio of 2, 3, or 10.

In the sixth lesson, the architect's problem was posed to students. In this problem, students were supposed to calculate different interesting magnitudes in a building. The base was either a rectangle (35 × 16) or a square (20 × 20) and the total area available for offices and apartments was known (5600 m^2). The height of each floor was given (3 m). The architect had to calculate the height of the building; the lateral area (that would be covered with glass); the total length of edges, except the base (aluminium devices were supposed to be fixed along the edges); and finally the volume (for the heating capacity). The aim of this problem was to oblige students to differentiate distinct kinds of perimetric, volumetric, and area magnitudes. The main obstacle for students was that the building's capacity (volume, in a way) was actually given through the intermediary of the total area available for apartments and offices. As a consequence, the number of floors was sometimes interpreted as the height of the building. The calculation of the lateral area was not very easy either, and some students tried to calculate it directly with the basic area or the total area available. The question about edges was, as expected, the easiest one. As for the volume, most groups did find it, but only one of them used the synthetic information: area available, 5600 m^2; height of each floor, 3 m. The other groups used the formula

$$L \times W \times H.$$

The last lessons were devoted to the study of triangular prisms, all having the same height and different base areas. In the first phase, students had to cut different bases from pieces of cardboard (8 cm or 16 cm wide) and to predict (before making them) the ranking order of the prisms built on these bases (six different bases, including equivalent-area bases). The prisms were then made, and sand was used to check the predictions. In order to explain the differences and the equivalencies, the concept of basic area was analyzed. To double the basic area, one can use twice the same area, or double the height of the triangle, or double the base of the triangle. A double-dependence table was drawn for the area of triangles and then for the volume of prisms (Schemas 5.30a and 5.30b).

(a) area of the triangle

		Height 4	8	12	16
Base	3				
	4				
	6				
	12				

(b) volume of the prism

			20	30	40
Basic area	16				
	24				
	32				

SCHEMA 5.30

Students worked on the first table and on the formula $A = 1/2\, BH$ by filling different cases. Their attention was drawn to the proportion between area and height (when base is held constant) and between area and base (when height is held constant). There are different possible procedures for the calculation of unknown cells: either the use of the formula or the application of adequate operators to previously known cells. Similar work was done on the second table.

Although all of these situations and behaviors deserve more detailed descriptions and explanations to be thoroughly understood, this chapter on multiplicative structures would have been very incomplete if this type of didactic experiment had not at least been briefly reported. Other didactic experiments are discussed in two previous papers (Rouchier, 1980; Ricco *et al.*, 1981; Vergnaud *et al.*, 1979).

Further Analysis and Experiments

As discussed in the first part of this chapter, problems met in ordinary economical and technical life involve different kinds of magnitudes and different categories of relationships. Asking students to solve such problems requires them to use logically distinct but psychologically interdependent topics in mathematics. Studying these topics in isolation is psychologically artificial. This is not specific to multiplicative structures: most didactic situations are conceptually pluridimensional. Three complementary approaches seem to be essential in this case: (*a*) fractions, ratios, and rational numbers, (*b*) linear and n-linear function with dimensional analysis, and (*c*) vector spaces.

Fractions, Ratios, and Rational Numbers

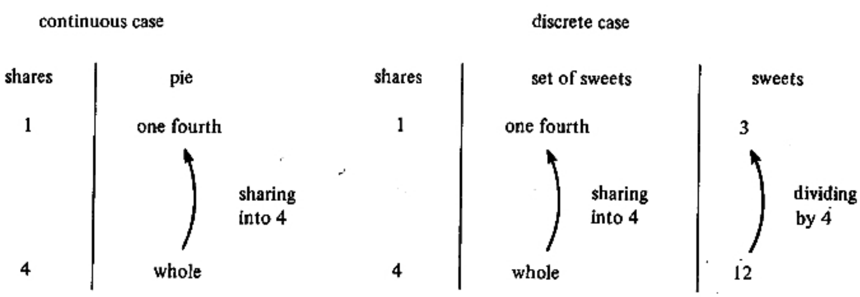
It is clear that multiplicative structures, because they imply multiplications and divisions, can be analyzed in a way that leads to fractions, ratios, and rational numbers. The main problem for students is that rational numbers are *numbers* and that entities involved in multiplicative structures are not pure numbers but measures and relationships.

The concept of rational number is defined, in mathematics, as an equivalence class of ordered pairs of whole numbers. This is a late construction in the history of mathematics. Fractions and ratios are not so well defined: the word *fraction* is sometimes used for a fractional part of a whole, sometimes for a fractional magnitude (that cannot be expressed by a whole number of units), sometimes for an ordered pair of symbols p/q, and sometimes for a relationship linking two magnitudes of the same kind. *Fraction* is rarely used for a relationship linking magnitudes of different kinds; the words *ratio* and *coefficient* are preferred. But *ratio* is also used for a relationship linking magnitudes of the same kind, and for an ordered pair p/q.

It would be futile to try to standardize the vocabulary. But as students' conceptions of rational numbers necessarily come from their conceptions of fractions and ratios, it is important to try to sort them out, in the light of our first analysis of multiplicative structures. Although decimals are very important in the development of rational-number concepts, I will not devote any special attention to them here. Brousseau (1980; 1981) and Douady (1980) have made very interesting contributions to the study of decimals.

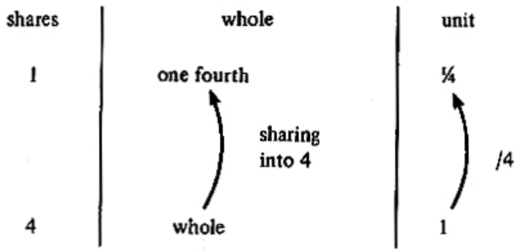
THEORETICAL CONSIDERATIONS

Sharing a whole into parts is undoubtedly the very first experience with fractions. It involves a direct proportion between shares and the magnitude to be shared (isomorphism of measures). This magnitude can either be discrete (a set of sweets, a packet of playing cards) or continuous (a pie, a sheet of paper, a bottle of lemonade). An important difference is that one does not usually know the measure of the continuous magnitude to be shared (neither the weight nor the area of the pie) whereas a discrete magnitude can usually be counted. Consequently, the unit value (one person's share) is necessarily expressed as a fractional quantity (one-fourth, one-sixth) in the continuous case, whereas it may also be expressed as a number of elements (three sweets each) in the discrete case. (See Schema 5.31.)



SCHEMA 5.31

The first type of division, mentioned in the description of the isomorphism of measures structure above, can easily be recognized in the discrete case (first and last columns) but it is not so easily recognized in the continuous case, or when one considers the discrete set as a whole (second column). In these latter cases, children have to recognize that sharing a whole into four parts, for example, requires dividing the unit 1 by 4. (See Schema 5.32.)



SCHEMA 5.32

This is a problem for elementary school children. Whereas whole numbers can be directly associated to quantities by counting, fractions (even the most elemen-tary ones, $1/n$) cannot be associated directly to quantities; they are relationships between two quantities. This conceptual difficulty may be different for continu-ous and discrete quantities, and for different numerical values, as we will see later.

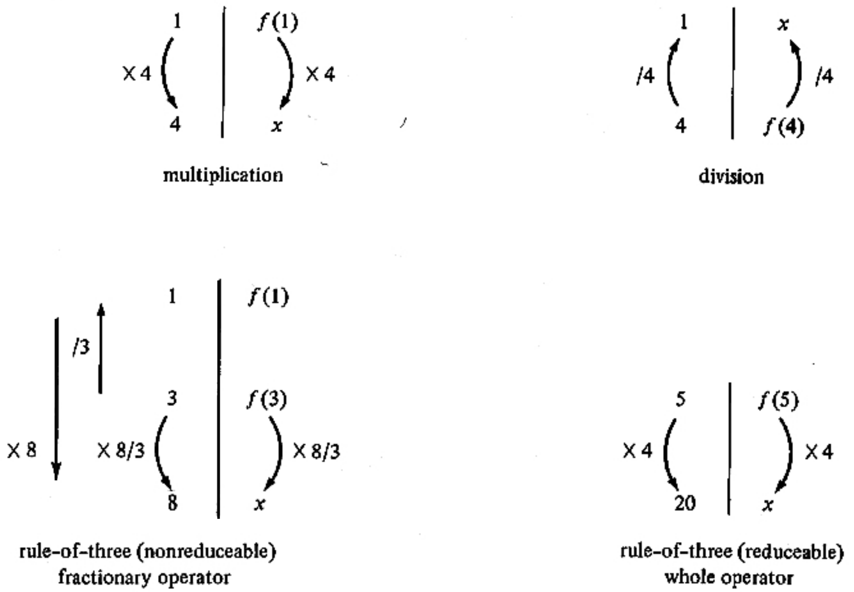
Once unit fractions have gained meaningfulness for children through sharing operations, many further steps remain for thorough understanding. We describe these steps in terms of non-Archimedean fractions p/q, smaller or bigger than 1, and equivalence relationships, and operations. The main conceptual problem for students is that fractions can be quantities, or scalars, or functions, and that these different concepts have to be integrated into one synthetic mathematical concept: the rational-number concept.

Because fractional quantities and magnitudes cannot be conceived without the help of scalar operators, I will start with this concept. As a scalar operator, a fraction links two quantities of the same kind. Being a quotient of two quantities of the same dimension, expressed in the same unit, it has no dimension and no unit. The problem of linking two quantities is raised either in comparisons (which quantity is bigger and how much bigger is one than the other?) or in proportional reasoning, as seen in the preliminary analysis above. Actually, comparisons do not necessarily involve fractions and ratios; differences (ob-tained by subtraction) are also appropriate in many comparison situations. In proportional reasoning, students must move from an additive method of compar-ing to a multiplicative one. Incidentally, this probably explains the well-known additive error in proportion tasks (Karplus & Peterson, 1970; Lybeck, 1978; Noelting, 1980a,b; Piaget, Grize, Szeminska, & Bang, 1968) and some other sophisticated errors (see error S′ in the rule-of-three experiment described above).

Schema 5.33 shows examples of scalar operators (whole and fractionary) from the simple multiplication and division cases to more complex cases (reducible and nonreducible ones).

There are two categories of ratios in comparison and proportion problems: either one quantity is part of the other (inclusive case) or there is no obvious inclusion relationship (exclusive case).

Scalar operations (whole and fractionary)



SCHEMA 5.33

1. Inclusive examples: p out of q. Peter ate two-fifths of the sweets. Three-fourths of the marbles are blue.
2. Exclusive examples: p to q. Peter's collection of miniature cars is three-fourths of John's collection. The distance covered in 5 hours is five-thirds of the distance covered in 3 hours.

There are some important differences between these categories:

1. Inclusive fractions or ratios are always smaller than 1 (except when the magnitude or the set is compared to itself), whereas exclusive fractions or ratios can be greater than, less than, or equal to 1.
2. Inclusive fractions are never reducible to whole numbers (except to 1 in the trivial case), whereas exclusive fractions can be either reducible or non-reducible (see examples in the previous schema.
3. Inclusive fractions have no inverse (for young students) because it is meaningless to consider the whole as a fraction of the part, whereas exclusive fractions have ''natural'' inverses: if Peter's collection is three-fourths of John's, John's collection is four-thirds of Peter's.
4. Inclusive fractions can be made meaningful to young students either by sharing operations, or more generally by subset–set proportions, whereas exclusive fractions necessarily involve comparisons.

Most children, at the end of elementary school, are unable to conceive exclusive fractions as fractions. Their model is the inclusive fraction model. This is a problem because comparisons and ratios between any two quantities of the same kind are a more powerful model than inclusive fractions, providing a more general foundation for scalar operators or ratios.

Fractional measures result from the application of fractional operators to other measures considered as wholes or units, in the nonreducible case. For example:

1. 3/10 cm is a fractional measure that results from dividing 1 cm into 10 parts and taking 3 parts, or else taking three-tenths of 1 cm.
2. 5/3 of 205 kms (1025/3) is a fractional measure of the distance covered in 5 hours, that results from applying 5/3 to the distance covered in 3 hours.

Nonunit fractional measures (i.e., p/q), like unit fractional measures (i.e., $1/n$), are necessarily relationships to other measures, via scalar ratios. But fractional measures can be added and subtracted, whereas scalar operators can only be composed in a multiplicative fashion.

Fractional scalar operators p/q are themselves the *concatenation* of one division by q and one multiplication by p, and all fractional scalar operators can be concatenated and composed into one single fractional scalar operator:

$$(\times\ p/q) \circ (\times\ p'/q') = (\times\ pp'/qq')$$

Addition and subtraction of fractional scalar operators are nearly meaningless.

The situation is not quite symmetrical with fractional measures, because they can be, most obviously, added and subtracted, but they can also be multiplied (or divided) by one another, when the structure is a product: "find the area of a rectangle that is 8/3 cm long and 4/7 cm wide."

Next, let us consider function operators. They are quotients of dimensions, and they raise conceptual difficulties for students. Nevertheless they provide the most natural way to introduce the concept of an infinite class of ordered pairs, as can be seen in Schema 5.34 taken from the wheat flour problem.

Kg of wheat	Kg of flour
1.2	1
12	10
18	15
24	20
30	25
$6n$	$5n$
6	5

SCHEMA 5.34

If 1.2 kg of wheat is used to make 1 kg flour, one can establish a correspondence table between 12 kg wheat and 10 kg flour, 18 and 15, 24 and 20, and so on. All ordered pairs $(6n, 5n)$ belong to this infinite class; the function from left to right can be expressed by any operator $\times$ 10/12, $\times$ 15/18. . . . The simplest operator is $\times$ 5/6 and (5,6) is the simplest element of the class of ordered pairs. It is also possible to exhibit the equivalence of scalar operators, but the infinite character of the class is not exemplified.

In conclusion, it appears that the meaning of concatenation of division and multiplication comes from fractional scalar operators and ratios, the meaning of addition and subtraction comes from fractional quantities, and the infinite character of each rational-number class comes from fractional function operators and ratios. Multiplication of rational numbers can be made meaningful through composition of scalar operators, composition of function operators, and even product of measures. The synthesis of all three aspects can occur only if measures, scalar operators, and function operators lose their dimensional aspects and the distinction between element and relationship, and if the concept of rational numbers as pure numbers is built up. But this concept inherits all three aspects. Teachers should not expect this construction to be easy, fast, or immediately understood by students. Students cannot work on meaningless objects and one should not be surprised when they try to make pure numbers meaningful by interpreting them as quantities or operators.

The above analysis is convergent with Kieren's (1978–1979–1980) analysis, although it was developed quite independently. The main originality of this analysis is that it is strongly related to the general framework of multiplicative structures, in a way that clarifies differences between quantities and operators and between scalar and function operators.

SCALAR VERSUS FUNCTION

The distinction between scalar and function aspects has been mentioned by other authors, but it may be very ambiguous in some situations. Freudenthal (1978) and Noelting (1980a; 1980b) have used the distinction interna l external; Lybeck (1978), the distinction within–between; and Karplus, Pulos, and Stage (this volume) have discussed findings on the preference of students for scalar or internal aspects rather than function or external aspects. I would like to stress a few points in order to clarify the main theoretical issues discussed.

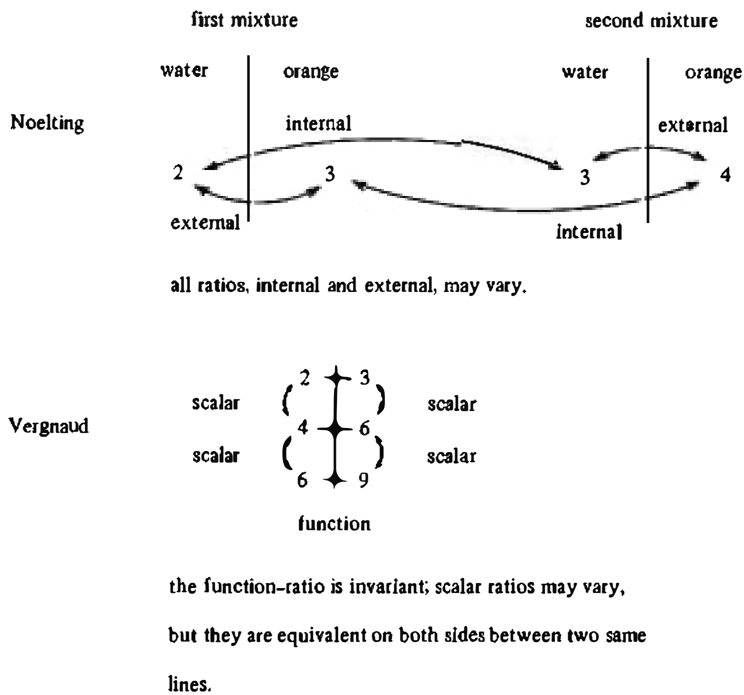
First, problems of comparing tastes, concentrations, or densities, as used in most studies on the development of ratio, (e.g., Noelting, 1980a; 1980b) are different from direct proportion problems (isomorphism of measures).

In direct proportion problems, there are only two variables and an invariant relationship (the function) between these two variables: the cost of goods, the speed, or the density is given as constant. The problem is to find $x = f(c)$, knowing a, $b = f(a)$, and c; it is not to compare two functions f and f'.

In most studies on ratio-concept development, the function is also a variable: for instance, in Noelting's excellent studies (1980a; 1980b) the experimental paradigm involves three variables: the number of glasses of water, the number of glasses of orange juice, and the taste or concentration, which is a quotient of the two other variables. The problem may be viewed in different complementary ways:

1. Direct comparison of two function-ratios
2. Decomposition of this problem into two other problems, relying upon two theorems:
   a. concentration is proportional to orange juice, provided water is held constant;
   b. concentration is inversely proportional to water, provided orange juice is held constant.

In this decomposition, one can easily recognize the structure of multiple proportion: the quantity of orange juice is a bilinear function of the quantity of water and of the concentration wished. Ratio-comparison situations cannot be analyzed as simple-proportion problems.



SCHEMA 5.35

Second, the dimensional analysis of function ratios is clear when M_1 and M_2 have different dimensions, such as: time in hours and distance in kilometers for speed, volume in cc and mass in g for volumic mass, water in glasses and sugar in spoonfuls for sugared-water concentration, or even when M_1 and M_2 are the same magnitudes, expressed in different units, as is the case in problems of conversion from one unit system into another one. In many studies on ratio-concept development, because experimenters want the unit to be the same (glasses of the same size for water and orange juice), the quotient of two dimensions does not appear as clearly. Moreover, it is not invariant. The internal–external distinction made by Noelting is different from the scalar–function distinction, as can be seen in Schema 5.35. Although some results on ratio-concept development are well established (Hart, 1981; Karplus *et al.*, Chapter 3, present volume; Noelting, 1980a; 1980b; Suydam, 1978), more experiments are needed to clarify the dependence of this development on the different aspects of multiplicative structures.

FIRST NOTION OF FRACTION

I now report some results on the scalar operator concept that have been obtained with a different experimental paradigm. In her dissertation on the first ideas of children on fractions, Mariam Salim (1978) has used three different tasks:

1. Knowing the referent quantity and the fraction operator, find the compared quantity
2. Knowing the referent and the compared quantity, find the fraction operator
3. Knowing the fraction operator and the compared quantity, find the referent quantity.

The tasks involved different sorts of magnitudes:

1. Discs (continuous) on which lines had been drawn, to divide them into n parts (2, 3, 4, 5, 6)
2. Strips of paper (continuous) with no drawing
3. Sets of pearls (discrete)

and different numerical values:

1. $1/N$ $(N < 10)$
2. P/Q $(P < 4\ Q < 10)$.

Her most important finding is that understanding fractions depends heavily on their numerical value. Whereas all three tasks are easily achieved with continuous and discrete quantities for 1/2 and 1/4, they are still very difficult for other fractions.

There is a four year *decalage* on most tasks between 1/2 and other unit fractions, except 1/4 which is intermediary, and 1/3 which is a bit more difficult than other unit fractions (in French *un tiers* does not refer to *trois* as simply as *un cinquième* does to *cinq*).

In the first grade (6 7-year-olds), 1/2 fractions are fully mastered by 30% of the students and partly mastered by 50%. In the second grade, a few students deal fairly well with 1/4 fractions and start dealing with other unit fractions. In the third grade, nearly 40% of the students reach that stage, but this proportion increases rather slowly in the fourth and the fifth grades. In the fifth grade, all students master 1/2 fractions but only 40% master 3/4 and 2/5 fractions.

The difficulty of the three tasks was expected to be different and also to vary for discrete and continuous quantities. Salim did find some differences, but not as large as she expected, and very small compared with the differences due to the numerical values. It seems to be typical of unit fractions that only slight differences exist between the "find the compared quantity" task, the "find the operator" task, and the "find the referent quantity" task. Differences are bigger for P/Q fractions, as we see subsequently.

If these findings were confirmed, it would be an argument in favor of the relational character of fractional quantities: either the three-term relationship is mastered in all three tasks, and then the fraction is understood, or none of the tasks is solved.

These findings must be contrasted with another result obtained by Salim in a simpler task: as students were presented discs (with drawn division lines) and asked to show one-half, one-quarter, one-fifth, — three-quarters, two-fifths, 50% of the first-grade students were successful in the most difficult items. But this success is ambiguous because the procedure used by students was counting. One typical error of young students in the "find the compared quantity" task (discrete case) illustrates this ambiguity: when asked to find one-fifth of a set of pearls, many young students made a subset of 5 pearls.

Another error illustrates the inclusive character of the very first notion of fraction for children: when asked to find one-fifth of a set of pearls, or a strip of paper, many students were satisfied with just *a part*.

Another interesting result was found by Salim in comparison tasks: except for 1/2 and 1/4 fractions, students were not able to compare $1/n$ and $1/n'$ fractions until the third grade. But first graders were able to say that 3/5 was bigger than 2/5 by using a model that has nothing to do with fractions: 3 "something" is bigger than 2 "something." Third graders and fourth graders were not as successful on this item, because they tried to take the fractional character into account (U-shape curve). Most interesting were the explanations given by students: when asked to compare $1/n$ and $1/n'$ fractions, younger students referred to whole numbers n and n' and failed, whereas older students (third and fourth graders) referred to the number of shares. These results support the thesis that the first notion of fraction is inclusive, and refers to sharing operations.

In another experiment (Vergnaud, Errecalde, Benhadj, Dussouet, 1979) with older students (fifth and sixth graders), we systematically compared the inclusive case and the exclusive case. In three different contexts (customers in a restaurant, trees in a forest, load of a truck), students were faced with the three tasks: find the compared quantity, find the fraction operator, find the referent quantity. All fractions were P/Q fractions (where P and Q were different from 1).

None of the tasks was trivial, but we did observe that the "find the compared state" tasks were easier than the two others (60% success instead of 40%).

Then we compared the "find the fraction operator" tasks in the inclusive case (comparison of the part and the whole) and in the exclusive case (comparison of one part to another part). In the exclusive case, the success rate was about half the rate obtained in the inclusive case: from 10 to 20% instead of 40%.

This last result has convinced me that it is necessary to study the ratio-concept development in different contexts and in different frameworks. It would probably be fruitful to plan an extensive study on multiplicative structures, with different experimental paradigms involving different aspects of the ratio concept.

Linear Function and *n*-linear Function: Dimensional Analysis

The second approach for developing multiplicative structures with students includes the concepts of variable, function, linear function, n-linear function, and dimensional analysis. Although a linear function is a mapping from the set of real numbers into itself, and not from rational measures into rational measures, the linear model fits very well with the multiplicative structures. A formal derivation has been attempted by Kirsch (1969). I do not repeat here the analysis described above. What I would like to stress is the necessity of identifying very clearly for students the different variables, the different operators, and the different ways of solving the same problem.

One way to clarify these distinctions is to use symbolic representations that discriminate among different variables, different relationships, and different operations. For example, representing data and solutions in tables helps discriminate magnitudes of different dimensions (different columns or lines for different kinds of magnitudes) and relationships of different types (scalar relationships, function relationships, inverses, composed scalar and function relationships). We have used such representations in different situations (Rouchier, 1980) and I will report here only one example used with eighth graders.

THE FARM PROBLEM

"A farm, in the Beauce country, has an area of 254.5 ha. Half of it is devoted to growing wheat. The average crop is 6800 kg of wheat per ha. One needs 1.2

kg grain to make 1 kg flour; 1.5 kg flour to make 4 loaves. One loaf is, on the average, the daily consumption of two persons."

Several tasks have been proposed to students:

1. Formulate and discuss a variety of questions. Are they well and completely formulated? Which ones are the same, under different formulations?
2. Make a table to represent the data and the relevant questions, and organize spatial correspondences.
3. Represent line-to-line and column-to-column operators.
4. Express the dimensional characteristics of these operators.
5. Explain the different solutions used by different groups of students for the same question and analyze them.
6. List the class of ordered pairs between any two columns (see the example of the wheat–flour function above) and the simplest fractional operator.
7. Identify $f(1)$ with the corresponding function operator.
8. Express and explain rules for composing column-to-column operators.

I can only summarize some experimental findings. More details can be found in Rouchier (1980). First, many questions are incompletely formulated and ambiguous. The equivalence of two different formulations is not immediately recognized. Second, the use of different columns and lines for different dimensions and for values that are not source and image of each other is not easily discovered by students. Once in use, however, the spatial organization of data and questions helps to clarify relevant relationships and calculations. The discrimination between simple proportion and multiple proportion can be more easily perceived and analyzed. Finally, function operations are difficult, except in the simple multiplication case.

Schema 5.36 provides examples of possible tables and questions.

area	wheat	flour	loaves	persons in one day
1	6800			
	b		1	2
	1.2	1		
		1.5	4	
127.5	*a*			c

Examples of questions

a What is the crop of the farm?

b How much wheat is needed to make one loaf?

c How many persons can one feed during one day with the crop?

d How much wheat is needed to feed 100,000 persons during 1 week?

		persons 100,000	c	
	1		a	
days	7	d		
	365			wheat

SCHEMA 5.36

One can easily imagine, in the first table, the succession of functions, with their dimensional meaning.

× 6800	× 1'/1.2	× 4/1.5	× 2
wheat/area	flour/wheat	loaves/flour	persons/loaf

Interesting tasks can be organized around inversions and compositions of these functions.

Vector space

The vector-space model is again too strong for the sort of situations described here. It would be misleading to develop with seventh and eighth graders a formal presentation of vector-space theory. Yet the distinction between measure spaces, scalar operators, and function operators is directly related to vector-space theory: for students, measures behave as vectors, scalar operations as linear combinations, and functions as linear mappings. Actually, measure spaces are only semivector spaces (measures are positive), scalars are only rational numbers, and measures are only one-dimensional vectors.

It is nevertheless possible to make students confront less trivial vector spaces and linear mappings. For example, suppose a restaurant buys 4 different sorts of fruit every day: x_1 kg grapes, x_2 kg peaches, x_3 kg pears, x_4 kg apples. Two other important variables are the total weight y_1 and the total cost y_2. If, for a period (a week, for instance), the cost of each sort of fruit is constant (a_1, a_2, a_3, a_4 per kg), there is a nice nontrivial linear mapping from (x_1, x_2, x_3, x_4) vectors into (y_1, y_2) vectors that make certain calculations easier. As

$$y_1 = x_1 + x_2 + x_3 + x_4$$
$$\text{and } y_2 = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4,$$

all isomorphic properties can be used and explained.

$$f(V + V') = f(V) + f(V')$$
$$f(\lambda V) = \lambda f(V)$$
$$f(\lambda V + \lambda' V') = \lambda f(V) + \lambda' f(V')$$

Conclusion

One might think that there is a major contradiction in this chapter between some experimental results showing the difficulty for students of multiplicative structures and some theoretical developments. Actually, these developments show that problems met by students, even at an early stage of their school curriculum, involve complex structures and concepts. This complexity is inescapable. How should we deal with it in mathematics education?

I have not referred to the Piagetian formal stage of development because I cannot see where one could trace the limit between a concrete stage and a formal stage in the development of such a diverse conceptual field. For instance, contrasting concrete numbers and pure numbers would be an oversimplification. It would be misleading to view measures as concrete numbers, or handling of operators as a concrete stage of understanding numerical operations; there are important formal ideas and theorems about measures and operators.

It is true that most general properties of rational numbers cannot be expressed and explained unless rational numbers are viewed as pure numbers. This is the case of cross-multiplying, for instance:

$$a/b = c/d, \qquad ad = bc, \qquad a/c = b/d$$

What would it mean to multiply a and d if a were an M_1-measure and d an M_2-measure? But it is true and important that understanding multiplicative structures does not rely upon rational numbers only, but upon linear and n-linear functions, and vector spaces too.

Many mathematics teachers have the illusion that teaching mathematics consists of presenting neat formal theories, and that when this job is well done, students should understand mathematics. In fact, concepts develop by problem solving, and this development is slow. Problem-solving situations that make concepts meaningful to students may be far removed from an advanced mathematical point of view. They are nevertheless essential and they must be carefully and completely analyzed so that the development of concepts may be traced and mastered.

Another tempting and inappropriate attitude is postponement: wait until students have reached a certain stage. This may be misleading too; there is no reason why students would develop complex concepts if they do not meet com plex situations. The pedagogical illusion (teach it properly, they will know it) and the natural development fallacy (wait until they reach the stage) are Scylla and Charybdis obstacles in mathematics education. The framework of conceptual fields, which provides teachers with a variety of situations and different-level analyses, should help them to make students progress, slowly but operationally. Still, there is a long way to go before we fully understand the development of multiplicative structures.

References

Most references to American and English literature can be found in other chapters in this volume, especially in Behr *et al.*, Lesh *et al.*, and Karplus *et al.* One finds below only French references and a few specific references in English.

Brousseau, G. Problèmes de l'enseignement des décimaux. *Recherches en didactique des mathématiques,* 1980, *1,* 11–58.

Brousseau, G. Problèmes de didactique des décimaux. *Recherches en didactique des mathématiques,* 1981, *2,* 37 127.

Douady, R. Approche des nombres réels en situation d'apprentissage scolaire: Enfants de 6 à 11 ans. *Recherches en didactique des mathématiques,* 1980, *1,* 77–111.

Freudenthal, H. *Weeding and sowing. Preface to a science of mathematical education.* Dordrecht. Boston: Reidel, 1978.

Hart, K. *Children's understanding of mathematics* (Vol. 11–16). London: Murray, 1981.

Karplus, R., & Peterson, R. W. Intellectual development beyond elementary school: Ratio, a survey (Vol. 2). *School Science and Mathematics,* 1970, *70,* 813 820.

Kieren, T. E. The rational number construct—its elements and mechanisms. In T. E. Kieren (Ed.), *Recent Research on number learning.* Columbus, Ohio: ERIC/SMEAC, 1980.

Kieren, T. E., & Nelson, D. The operator construct of rational numbers in childhood and adolescence—An exploratory study. *The Alberta Journal of Educational Research,* 1978, *24,* 22 30.

Kieren, T. E., & Southwell, B. The development in children and adolescents of the construct of rational numbers as operators. *The Alberta Journal of Educational Research,* 1979, *25,* 234 247.

Kirsch, A. An analysis of commercial arithmetic. *Educational Studies in Mathematics,* 1969, *1,* 300–311.

Lybeck, L. *Studies of mathematics in teaching of science in Goteborg.* Goteborg, Institute of Education, 1978, No. 72.

Noelting, G. The development of proportional reasoning and the ratio concept: Differentiation of stages (Part 1). *Educational Studies in Mathematics,* 1980, *11,* 217 253. (a)

Noelting, G. The development of proportional reasoning and the ratio concept: Problem structure at successive stages; problem-solving strategies and the mechanism of adaptive restructuring (Part 2). *Educational studies in Mathematics,* 1980, *11,* 331–363. (b)

Piaget, J., Grize, J. B., Szeminska, A., & Bang, V. *Epistémologie et psychologie de la fonction.* Paris, Presses Universitaires de France, 1968.

Ricco, G. *Le développement de la notion de fonction linéaire chez l' enfant de 7 à 12 ans.* Thèse de 3ème cycle, Travaux du Centre d'Etude des Processus cognitifs et du Langage, 1978, No. 11.

Ricco, G., & Rouchier, A. Mesure du volume: Difficultés et enseignement dans les premières années de l'enseignement secondaire. *Proceedings of the Fifth Conference of the International Group for the Psychology of Mathematics Education, Grenoble,* 1981, 114–119.

Rogalski, J. Quantités physiques et structures numériques. Mesure et quantification: Les cardinaux finis, les longueurs, surfaces et volumes. *Bulletin de l'Association des Professeurs de Mathématiques de l'Enseignement Public,* 1979, *320,* 563–586.

Rogalski, J. Acquisition de la notion de dimension des mesures spatiales de longueur et surface. *Proceedings of the Fifth Conference of the International Group for the Psychology of Mathematics Education, Grenoble,* 1981.

Rouchier, A. Situations et processus didactiques dans l'étude des nombres rationnels positifs. *Recherches en Didactique des Mathématiques,* 1980, *1,* 225–276.

Salim, M. *Etude des premièes connaissances sur les fractions chez l'enfant de 6 à 12 ans.* Thèse de 3ème cycle. Paris, 1978.

Suydam, M. N. Review of recent research related to the concepts of fraction and ratio. *Proceedings of the Second Conference of the International Group for the Psychology of Mathematics Education, Osnabruck,* 1978.

Vergnaud, G. Didactics and acquisition of "multiplicative structures" in secondary schools. In W. F. Archenbold, R. H. Driver, A. Orton, & C. Wood-Robinson (Eds.), *Cognitive development research in science and mathematics.* Leeds: University of Leeds Press, 1980.

Vergnaud, G. *L'enfant, la mathématique et la realite.* Berne: Lang 1981.

Vergnaud, G. A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser, & T. A. Romberg, (Eds.), *Addition and subtraction: A cognitive perspective.* Hillsdale, New Jersey: Erlbaum, 1982.

Vergnaud, G. Cognitive and developmental psychology and research in mathematics education: Some theoretical and methodological issues. *For the Learning of Mathematics.* 1982, 3, 2, 31–41.

Vergnaud, G., Ricco, G., Rouchier, A., Marthe, P., & Metregiste, R. Quelle connaissance les enfants de sixièe ont ils des structures multiplicatives élémentaires? *Bulletin de l'Association des Professeurs de Mathématiques,* 1978, *313,* 331–357.

Vergnaud, G., Rouchier, A., Ricco, G., Marthe, P., Metregiste, R., & Giacobbe, J. *Acquisition des structures multiplicatives dans le premier cycle du second degré.* RO No. 2, I.R.E.M. d'Orléans, Centre D'Etude des Processus cognitifs et du Langage. 1979

Vergnaud, G., Errecalde, P. Benhadj, J , & Dussouet, A. La coordination de l'enseignement des mathématiques entre le cours moyen 2ème annèe et la classe de sixième. *Recherches Pèdagogiques,* 1979, No. 102.