Theoretical frameworks and empirical facts in the psychology of mathematics education

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PLENARY ADDRESS:

THEORETICAL FRAMEWORKS AND EMPIRICAL FACTS IN THE PSYCHOLOGY OF MATHEMATICS EDUCATION

Gérard Vergnaud (France)

Psychology of mathematics is not really a new field of research, as several psychologists started to investigate that field a long time ago. But it has developed and changed a lot during the last 15 years. The international group "Psychology of Mathematical Education", which is a working group of the International Congress on Mathematical Education, has played an important part in gathering ideas and new results together. Let me mention the name of Ephraim Fischbein who launched that working group and was its first president.

The aim of my lecture is to present in a synthetic fashion some of the results that have been obtained and some theoretical ideas which, I think, are essential to understand how students learn mathematics and develop their own ideas, and how teachers can improve their teaching, by taking account of the way students learn, or fail to learn.

Many mathematicians think that primary school mathematics is not mathematics, because there is nothing like proof in it, nor does it include any sophisticated concepts like those of function, continuity, or algebraic structure. This is a wrong view, and my first task will be to give examples of the conceptual importance, from a mathematical point of view, of children's first acquisitions.

Another task will be to show that mathematical concepts and procedures are learned and developed over a long period of students' growth, more than 10 years for additive structures or for multiplicative structures.

My next purpose will be to analyse the role of symbols and language in concept formation and problem solving, and to illustrate the specific, and sometimes difficult, operations of thinking that are required by the reading and the use of mathematical symbolisms like graphs and algebra.

And finally I will stress the need for a better interaction of epistemology of mathematics, cognitive, developmental and social psychology, and didactics.

What is a mathematical behaviour?

There are many kinds and many levels of mathematical behaviour, even in school mathematics. It is not easy to compare such behaviours as counting a set of discrete objects, measuring the length of a room, or the volume of a container, perform a difficult subtraction, choose the data and the operations that are necessary to solve a double proportion problem, a linear equation or a linear system, analyse the variation of a function, show the equality of two angles in a complex geometrical figure.

And yet it is important to consider that all these behaviours are tied to mathematical concepts, and would not be possible if such concepts as those of one-to-one correspondence, cardinal, additivity of measures, place-value notation, isomorphism of measures, linearity and n-linearity, function and variable, did not exist.

The fact that such concepts exist does not mean that students are fully aware of the relationship between these ideas and the way they behave. Most often mathematical ideas are only implicit in students' behaviour. Let me take two examples:

The first one is inspired by the work of Gelman and Gallistel (1978), Fuson and Hall (1982), Steffe et al. (1983), Chichignoud (1986) and others. When counting a set of six elements, most five or six year-olds count: one, two, three, four, five, six... six! Not only do they have to establish one-to-one correspondences between the objects to be counted, the gestures of the finger, the movements of the eyes and the number-words, but also they feel the need to say the word "six" twice. The first utterance refers to the sixth element of the set, the second utterance refers to the cardinal of the set: this double utterance means that the concept of cardinal has been recognized.

Older children may not repeat "six", but only stress it differently from the other number-words. But there are children who do not repeat "six" and when asked how many objects there are, are unable to answer, and start counting again.

For children who can cardinalize, "six" summarizes the information on the set gathered by the counting procedure. This is not the case for children who do not cardinalize. One can therefore infer the existence and the non-existence of the concept of cardinal from children's behaviour. Actually rather than a concept it is an invariant property of discrete sets relying upon an invariant organization of the counting behaviour. Let us call such invariant organization of behaviour, for a certain class of situations, a "*scheme*".

One can see from this example that the scheme of counting a set is not only made of rules of production but also of implicit strong mathematical ideas, namely those of one-to-one correspondence and cardinal. Let us call these ideas "operational invariants". Children progressively strengthen and extend the scope of validity of such invariants: cardinals do not depend on the spatial distribution of the elements, nor on the size of objects... And the counting procedure is altogether the same, even if it is more difficult to count a flock of moving sheep than a pile of plates on the table.

My second example will be taken from algebra. When dealing with equations like

$$4x + 25 = 53 \quad \text{or} \quad 41 = 3t + 26$$

many 13–14 year olds in France can apply the same scheme (invariant organization of the action sequence), for instance:

$$\begin{array}{ll} 4x + 25 - 25 = 53 - 25 & \qquad 41 - 26 = 3t + 26 - 26 \\ 4x = 28 & \qquad 15 = 3t \\ 4x/4 = 28/4 & \qquad 15/3 = 3t/3 \\ x = 7 & \qquad 5 = t \end{array}$$

or simplified versions like:

$$\begin{array}{ll} 4x = 53 - 25 & \qquad 41 - 26 = 3t \\ 4x = 28 & \qquad 15 = 3t \\ x = 28/4 & \qquad t = 15/3 = 5 \end{array}$$

The same scheme applies to all equations of the shape $ax + b = c$ whatever the name and the place of the unknown may be, provided a, b and c are positive, b is smaller than c, and a is a small whole number, smaller than $c - b$. Why these restrictions? Because for many students, the scheme cannot be fully extended to negative numbers and to difficult divisions.

Yet the procedure has some generality and it relies on such mathematical ideas as the conservation of equality when the same number is subtracted from both sides, or when both sides are divided by the same number. For many students these ideas are only implicit theorems. Let us call them theorems-in-action. They are operational invariants, like the ideas of cardinal and one-to-one correspondence.

I can give quickly a few more examples of implicit concepts and implicit theorems that can be traced in the emergence of new competences in children.

It is now well known for instance that when they have to count a set of children after having counted 4 boys and 3 girls, many five or six year-old children count the whole set again. It is a big step to be able to say $4 + 3 = 7$ or even to start from four, and count three steps forward. It shortens the counting-all procedure. This discovery (it is a fresh discovery for students as nobody usually teaches it to children) can be considered as the spontaneous recognition of the fundamental axiom of the theory of measure:

$$\text{card}(A \cup B) = \text{card}(A) + \text{card}(B)$$
$$\text{provided } A \text{ and } B \text{ have no common part.}$$

Don't count the whole set again; just add the cardinals of the subsets.

It is also well known (Carpenter and Moser, 1983; Fuson, 1983) that young students tend to simulate as closely as possible the structure of problems. For instance, when they have to find the result of winning three marbles when the winner had 6 marbles before playing, they would proceed by starting from 6 and count 3 steps on. If the situation is a start of 3 and a win of 6, the natural tendency is to start from 3 and count 6 steps on. This is not so easy and the risk of going wrong is bigger; another step for 5 to 7 year olds, is to start from the greater number 6 and count 3 steps on. One can consider this discovery as commutativity in action:

$$3 + 6 = 6 + 3$$

$$a + b = b + a$$

The realm of validity of such a theorem-in-action is not as large as that of the real theorem. Moreover, children cannot usually explain clearly why it is possible to do $6 + 3$ instead of $3 + 6$, but the mathematical idea is nevertheless present.

My last example will be the spontaneous solution given by some students to double proportion problems. In a verbal problem students had to calculate the quantity of sugar necessary for 50 children going to a vacation camp for 28 days. They had found in a book that the quantity needed for 10 children for 1 week was 3.5 kg. Some students said that 50 children is 5 times more than 10, and 28 days 4 times longer than a week; therefore the consumption of sugar should be 20 times bigger. The non trivial theorem-in-action revealed by this procedure can be written:

$$\text{Consumption } (5 \times 10, 4 \times 7) = 5 \times 4 \times \text{ Consumption } (10, 7)$$

which is a particular case of the general property of bilinear function:

$$f(\lambda_1 x_1, \lambda_2 x_2) = \lambda_1 \lambda_2 f(x_1, x_2)$$

Of course the easy numerical values made it possible for 11 year-olds to extract ratios 5 and 4; and there is no conceptual difficulty for them in recognizing the double proportion of consumption to the number of persons and to the length of time. Therefore, there is a gap between the theorem-in-action they used and the general theorem. But again the mathematical idea was there.

In summary a mathematical behaviour is a behaviour that relies upon some mathematical idea. A mathematical behaviour consists of an invariant organization of behaviour called a scheme. Some schemes are algorithms, but not all mathematical schemes are algorithms; and even when students have been taught an algorithm, they don't always follow the rules they have been taught, and replace them by more meaningful schemes or by memorized shortcuts.

Schemes (et algorithms) are not made of rules only; they are also made of goals and expectations, inferences and operational invariants. I have given a few examples of such operational invariants: cardinal, additivity axiom, commutativity, conservation of equality, bilinear theorem. The implicit knowledge contained in schemes can be analysed as made of concepts and theorems that are used in action, without being clearly identified and worded as objects, properties and relationships.

Situations and conceptual fields

A scheme is associated with a class of situations. There are many different schemes because there are many different classes of situations. For instance it is not the same conceptual problem and not the same scheme to add 7 and 5 in the three following problems:

1. Peter had 5 marbles. He plays a game of marbles with John and wins 7 marbles. How many marbles does he have now?
2. Robert has just played a game of marbles with Celia. He has lost 7 marbles. He has now 5 marbles. How many marbles did he have before playing?
3. John has played two games of marbles. He has lost 7 marbles in the second game, but he does not remember what happened in the first game. When he counts his marbles in the end, he finds that he has altogether won 5 marbles. What happened in the first game?

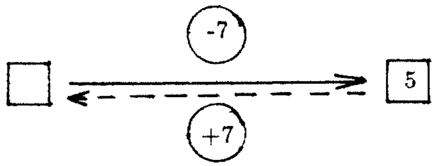
Problem 2 (Robert) is usually solved with a delay of one year and a half after Problem 1 (Peter) is solved. The main reason for this lies in the structure of the problems. Whereas Problem 1 (Peter) consists of searching the final state knowing the initial state and the transformation, Problem 2 (Robert) consists of searching the initial state knowing the final state and the transformation: this requires either the inversion of the transformation, or a hypothetical reasoning on the initial state such as: had he 10 marbles in the beginning, he would be left with 3; he needs 2 more; Robert had 12 marbles.

Let me represent both problems:

Problem 1 $\boxed{5} \xrightarrow{\quad +7 \quad} \boxed{}$

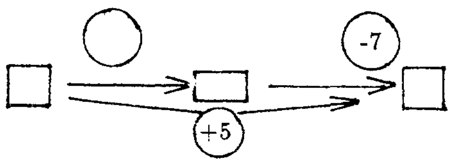
Problem 2 $\boxed{} \xrightarrow{\quad -7 \quad} \boxed{5}$

The addition 5+7 does not have the same meaning in both cases. In problem 1, it fits perfectly with the primitive conception of addition as an increase (Gelman and Gallistel, 1978). In problem 2, it goes against that primitive conception (you don't add when you have lost marbles), and it requires the inversion of the transformation -7, from the initial to the final state, into +7, from the final to the initial state.



Problem 3 (John) is the most difficult case; 75% of 12 year old students fail to solve that problem. Addition 5 + 7 is now totally counterintuitive, as you need to add a part and the whole to find the other part.

Problem 3



As a matter of fact the two games are considered as parts of the combined transformation which connects the initial and the final states. Had John lost 15 marbles in all, the answer would have been easy and natural. But problem 3 requires an addition 5 + 7, which is actually a subtraction of two directed numbers:

$$x + (-7) = (+5)$$
$$x = (+5) - (-7) = 5 + 7 = +12$$

The research work that has been achieved by Carpenter and Moser (1983), Riley, Greeno and Heller (1982), Vergnaud and Durand (1976), Nesher (1982) and others shows consistent results and substantial agreement about the classification of addition and subtraction tasks.

Whereas addition is usually conceived by mathematicians and teachers as the binary commutative combination of two parts into a whole, and subtraction as the search for one part knowing the whole and the other part, the psychological classification of cognitive tasks reveals that, beside the binary combination of two parts into a whole, there is also the unary operation of a transformation of the initial state. There are essentially six different tasks related to the initial-state — transformation — final-state relationship, among which two are solved by adding and four by subtracting. A similar situation exists for comparison problems, when you deal with three-term relationships: reference-state — comparison — compared-state, for instance:

John has 7 sweets, he has 2 sweets less than Janet. How many sweets does Janet have?

The binary commutative combination cannot model these relationships; one rather needs a unary non-commutative operation, as can been seen in the above arrow-diagrams.

Let me also stress the fact that it is not wise to keep directed numbers away from children at the primary school level, as they do meet situations involving transformations and relationships which should be modelled by directed numbers. I do not find as much support as I would like in favour of this idea, probably because some operations with negative numbers are still difficult for many 15 or 16 year-old students. But the fact that some tasks with negative numbers are difficult for the majority of 15-year olds must not hide that other fact that most 9 year-olds can understand what a negative transformation is (I have eaten four sweets), or what a negative relationship is (I owe you 3 dollars), and can also understand that negative and positive transformations are inverses of one another.

In summary, if one calls "additive structures" the set of situations that involve the addition or subtraction of two numbers, one sees some primitive competences and conceptions emerging in 3 or 4 year-old children; and still some problems requiring just one addition are failed by a majority of 15 year-olds. Between these two periods of their cognitive development, children discover or learn how to solve a great variety of problems.

Researchers are now able to give a differentiated picture of the variety of cognitive tasks which children meet, of the mental "revolutions" they have to achieve, of the main obstacles on which some students keep failing for a long time.

Several important concepts are involved in additive structures: cardinal, measure, state, transformation, comparison, difference, inversion and directed number are all essential in the conceptualizing process undertaken by students. Some of these concepts remain implicit for them. Understanding them sometimes requires the teacher to provide explicit wording, symbolizing and explaining.

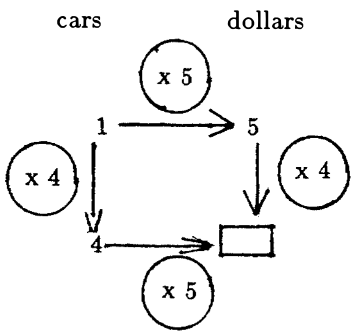
The place of language and symbols is certainly an important issue in mathematics education. I will come to this point later.

But before I do this, I would like to take other examples of the long term development of mathematical competences and conceptions. My next example will be "multiplicative structures", i.e. the set of situations that involve the multiplication or the division of two numbers, or a combination of such operations. Most of these situations are in fact simple-proportion or multiple-proportion problems, in which two variables are proportional to each other (simple proportion), or one variable is proportional to several other independent variables (multiple proportion).

Children are faced with proportion problems as soon as they have to find the cost of several identical objects, share a number of sweets, or find how many pastries they can buy with a certain amount of money.

In simple proportion problems, there are two kinds of ratio: scalar ratios (between magnitudes of the same kind) and function ratios or rates (between magnitudes of different kinds), sometimes called intensive quantities (Schwarz, 1988).

Take for instance a multiplication problem like the following: Miniature cars cost 5 dollars each. How much do you have to pay if you buy 4 of them?



The above diagram shows that you can either use the invariant scalar ratio (vertical):

$$\frac{\text{cost of 4 cars}}{\text{cost of 1 car}} = \frac{\text{4 cars}}{\text{1 car}}$$

or the invariant function ratio or rate (horizontal):

$$\frac{\text{cost of 4 cars}}{\text{4 cars}} = \frac{\text{cost of 1 car}}{\text{1 car}}$$

The first one is a scalar and has no dimension; it will be expressed, in natural language, by such expression as "4 times more". The second one is a quotient of two magnitudes "dollars per car" and it relates the two different variables; this is the reason why I call it a function-ratio.

If one introduces multiplication as repeated addition of the same number, it is meaningful to do this with the scalar operator in mind:

$$\text{4 times 5 dollars} \leftrightarrow \text{5 dollars} + \text{5 dollars} + \text{5 dollars} + \text{5 dollars}$$

but not with the function operator, as 5 times 4 cars cannot give dollars.

The correct analysis would be:

$$\text{4 cars} \times \frac{\text{5 dollars}}{\text{1 car}} = \boxed{} \text{ dollars}$$

Thus, students meet dimensional analysis at the primary school level.

Many results have been collected on the comparative difficulty of multiplication and division problems in the simple proportion case. One may vary the structure of the problem, the numerical values and the domain of experience to which such problems refer. See for instance Bell, Fischbein and Greer (1984), Vergnaud (1983).

multiplication		type I division case partition		type II division quotient		general case rule of three	
1	b	1	□	1	b	a	b
a	□	a	c	□	c	c	□

a, b, c can be taken among small or large whole numbers, among decimals, larger or smaller than 1, among fractions.

And the problems may refer to familiar or unfamiliar domains of experience, to easy or difficult mathematical physical or technical domains.

- Type II division is, on average, rather more difficult than type I division.
- Multiplying by a decimal is a striking obstacle, probably because it moves the child away from the primitive conception of multiplication as iterated addition.
- It is even more difficult to divide by a decimal.
- In children's primitive conception, multiplication is supposed to make bigger and division smaller. Therefore multiplying or dividing by numbers smaller than 1 gives counterintuitive results.
- It is also more natural for students to divide the larger number by the smaller. Many errors appear when the correct division is smaller by larger.
- The difficulty of rule-of-three problems is altogether tied to the numerical values. The procedures used by students favor the use of the isomorphic properties of the linear function

$$f(x + x') = f(x) + f(x')$$
$$f(\lambda x) = \lambda f(x)$$
$$f(\lambda x + \lambda' x') = \lambda f(x) + \lambda' f(x')$$

rather than the constant function coefficient:

$$f(x) = ax$$
$$x = \frac{1}{a} f(x)$$

This means that students consider scalar ratios, between magnitudes of the same kind, and linear combinations of such magnitudes, rather than function-ratios, between magnitudes of different kinds. Nevertheless the best students can shift easily from one point of view to the other, whereas the weak students keep using the same stereotyped strategy. These results are important for teachers, as some taught procedures are practically incomprehensible for most children, like the cross-product for rule-of-three problems.

Several researchers (Noelting, 1980; Karplus et al, 1983; Behr et al, 1983; Kieren, 1988) have studied extensively the difficult development of the concept of ratio. The synthesis into the concept of rational number of such different ideas as those of fractionalquantity, scalar ratio and function-ratio, is a long-term product of mathematical education.

But the most important conceptual problem in multiplicative structures, is probably the multiple proportion structure, which is involved in the measure of space (area and volume), in different domains of physics (quantity of movement, heat and energy, electricity...) and in many other domains like those of production and consumption, when some variable is proportional to two or more independent variables.

Let us take the example of the concept of volume. There is some understanding of that concept by 6 or 7 year-old students, when they have to compare the capacity of different containers, or when they have to measure, and even estimate the capacity of one container with the help of a unit of volume. Rogarski and I call unidimensional, the conception of volume that is sufficient to accomplish such tasks, and tridimensionnal the conception that is involved in the understanding of the concept of volume as the product of three lengths, or the coordination of ratios concerning lengths, areas and volumes.

In a task concerning the comparison of the capacity of two fish tanks (the large one was twice as long, three times as wide and twice as deep as the small one), we could observe several different procedures used by students, correct or incorrect (Vergnaud, 1983). I will take the example of two correct procedures:

1) $2 \times 3 \times 2 = 12$ times larger.
2) mentally pave the large tank with the small one: two in the length, three in the width; this makes 6; two in the height; 6 and 6 make 12.

The first procedure is tridimensional; it can be generalized to non-whole number ratios. The second procedure is unidimensional; it fails with ratios like 1.5, 0.8, 1.2, as one cannot easily pave the large tank with the small one.

One can also make different uses of the formula for the area of regular prisms:

$$V = A \cdot H$$

1) To calculate the volume, you must know the basic area and the height, and multiply.
2) You can also calculate the height (or the area) when you know the volume and the area (or the height). This reading of the formula is not contained in the first reading.
3) The volume is proportional to the area when the height is held constant, and proportional to the height when the area is held constant. This reading of the formula is actually the best way to understand it, and the true reason for it. But it is very rarely provided by schoolbooks, at least in France.
4) If you take units ten times smaller for lengths, the measure of the volume will be 1000 times bigger.

Students may be able to read the formula at a certain level and be unable to understand the tridimensional nature of the concept of volume, which is yet essential. And still they know something about the concept of volume.

It is for this very reason that I have developed the framework of conceptual fields: we need a framework to understand connections and jumps, in the development and the learning of competences and conceptions. The main organizer of such fields is the content of knowledge, and not such abstractions as the logical structure, the linguistic structure, or the level of complexity as measured by information theory. The complex conception of volume comes from the enrichment of former conceptions of volume, together with the interaction of these conceptions with new situations which require students to take account of new relationships, both spatial and multiplicative.

Additive structures are a conceptual field. Multiplicative structures are another conceptual field, not totally independent from the former one, but sufficiently independent to be studied separately. The conceptual field of multiplicative structures is made of such concepts as those of linear and n-linear functions, ratio, and rational number, dimensional analysis, vector-space... It involves situations of different kinds, taken in different domains, that can be analysed by simple and multiple proportion structures. It also involves different words and natural language expressions, different symbolizations like tables, graphs and formulas. It finally involves the implicit or explicit recognition by students of a variety of operational invariants like those mentioned above: scalar ratio, function ratio, independence and dependence of variables, theorems-in-action such as the isomorphic properties of the linear and the n-linear functions.

This framework is a critical means to understand how students learn: it claims that concepts are rooted in situations, consist of invariants of different kinds and levels, and need to be represented by linguistic and non-linguistic symbolic elements.

Algebra

I do not have the time to explain in detail the numerous problems raised by the learning and teaching of algebra. Convergent results have been collected on the way students shift, or have difficulties in shifting, from arithmetic to algebra, also on the errors they make in reading and transforming algebraic expressions, on the conceptual difficulties raised by the concepts of function and variable, and on the use that can be made of calculators and computers : Filloy, Rojano (1984, 1985), Booth (1984), Vergnaud et al. (1987), Kieran (1988).

I would also like to stress the important theoretical problem of the relationship between symbols and syntax on the one hand, mathematical knowledge and schemes on the other hand.

Algebra is not an independent conceptual field, that could be taught and learned independently of additive and multiplicative structures. Even if teachers and schoolbooks take it for granted that algebraic expressions are just relationships between numbers, students do not: they keep considering numbers as magnitudes (cardinals, lengths, areas, money, physical quantities...), and as relationships between magnitudes. Moreover, as they read expressions from left to right, they often consider that algebraic expressions model situations in which time is considered as going from an initial state on the left, to a final state on the right. In such a conception, the equality sign does not mean the symmetric and transitive equality relationship that is required to understand equations, but rather a relationship between a production process and an outcome.

Therefore the teaching of algebra, especially the introduction of algebra to 12 to 15 year-old students, requires a careful epistemological and cognitive analysis.

As algebra makes important use of syntactic rules, (you must do this, you may not do that), it is important to clarify the relationship between symbols and symbolic operations on the one hand, and the mathematical magnitudes, relationships and operations that are represented by those symbols on the other hand. Let me take four examples:

1) $3x + 12 = 6x - 3$
2) $3x^2 + 12x = 0$
   $3x(x + 4) = 0$
3) $(n - 1) + (n + 1) = 2n$
4) $(a + b)(a - b) = a^2 - b^2$

The meaning of example 1 is usually: find the value of x so as both functions $3x + 12$ and $6x - 3$ have the same value. The equality concerns numbers not functions. It is true only for x correctly chosen. The meaning of example 2 is somewhat different. It is aimed at showing that 4 and 0 are solutions. The meaning of example 3 is again different. It demonstrates that the sum of the predecessor and the successor of any whole number is an even number. The equality concerns predecessor and successor as functions of any given number, and the equality is always true. This is also the case in example 4. The meanings of elementary algebra are manifold.

Algebra is usually introduced as a way to solve arithmetic problems.This is usually viewed by students as more complex than an arithmetic solution, because in most simple equations $a + x = b, ax = b$ and $ax + b = c$, the algebraic solution actually depends upon the arithmetic solution and does not offer any obvious benefit.

To appear as a profitable tool, algebra must be seen as a way to solve arithmetical problems that cannot be solved easily by purely arithmetical means. Researchers have studied problems of this type. They have usually arrived at problems that can be expressed either as

$$ax + b = cx + d$$

unknown on both sides or as

$$\begin{aligned} ax + by &= c \\ a'x + b'y &= c' \end{aligned}$$

two unknowns.

To solve these equations, students must accept operating upon the unknowns. This is just what some of them don't understand. How can you operate on something or with something you don't know? (Collis, 1975)

Another difficult cognitive problem concerns negative numbers. Students' most frequent conception is that numbers are magnitudes; they cannot be negative. The minus sign means subtraction of a positive quantity, not the inversion of a transformation or a directed number, nor a difference between two transformations or two relationships. With such conceptions what does it mean to find a negative solution?

Here again one needs to find problems that make negative solutions meaningful: problems in which unknowns are transformations, relationships, or coordinates.

When one observes students solving equations, one can usually identify organized and standardized patterns of behaviour. Automaticity is a powerful property of algebra. But automatic algorithms are only the visible part of the iceberg. The profound ideas of algebra need to be clarified and this cannot be done without the identification of the concepts of function and variable, of directed number, and so on. Not only must algebra be a useful tool, it consists also of new objects, the epistemological status of which is not clearly seen by students and even by teachers.

The meaning of the concept of function and variable is not conveyed by equations only, and not even essentially, as letters are conceived of as unknowns rather than variables. Therefore computers and programmable pocket calculators, graphic curves and other devices are essential ways for students to experience functions and variables and identify them as mathematical objects, having names, properties, and relationships to other objects.

Symbols and mathematics

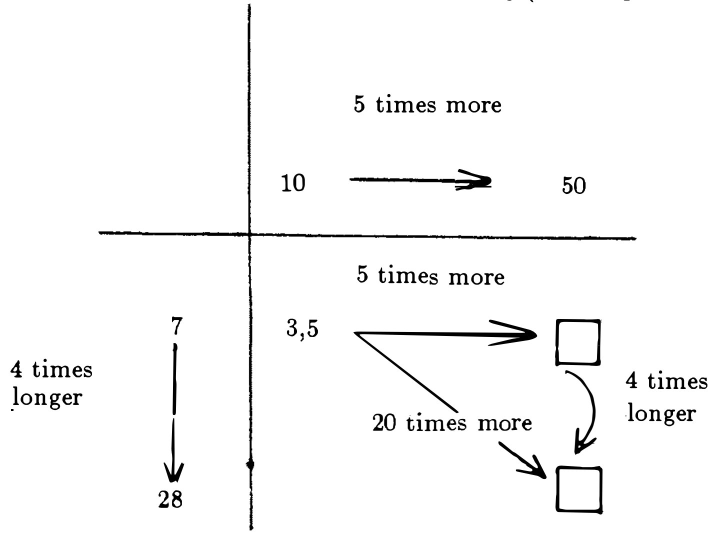
The role of symbols can therefore be clarified. Symbols are necessary to identify mathematical objects and make explicit their properties and their relationships to other objects. Whereas in the arithmetic solution of a problem, one very often leaves implicit the choice of the relevant data and operations, the algebraic solution consists of making the relationships explicit and summarized in a laconic expression; then there are algorithmic or quasi-algorithmic ways of dealing with these expressions.

Actually the role of symbols in thinking is a very old psychological and even philosophical problem. Vygotski developed strong ideas about this problem more than 50 years ago, when he studied the relationship between language and thinking.

Research on mathematics education makes use of and is in tune with some of his ideas. Natural language and mathematical symbols of all kinds (tables, diagrams, graphs, algebra...) play an important part in the process of conceptualizing, also in the control and regulation of schemes et algorithms, also in the solving of new problems, and in reasoning about them, i.e. combining and transforming relationships, planning, choosing data and operations.

Algebra is certainly the most obvious case in school mathematics, of the help of symbols in thinking, but there are many earlier cases, at the elementary school level with diagrams and tables, and even at the kindergarten level, when children start counting with words, making a specific use of natural language in counting, in counting on and down, in counting up to a certain number from a given number, in adding, subtracting and comparing, also in expressing spatial relationships and movements.

If I may choose just another example of the help of symbols in thinking, I will take the case of the above-mentioned bilinear reasoning (consumption of sugar).



5 times more children and 4 times more time make 20 times more consumption of sugar.

But graphs and diagrams also help thinking a lot. For instance graphs are a powerful tool in algebra, to represent variables, functions and solutions of systems during the introductory phase of algebra.

But a new symbolic tool like graphs does not go without difficulties. The reading, understanding and use of such a tool raises specific conceptual difficulties.

For instance, it is not obvious that numbers can be represented by dots on a line. It requires the understanding of the concept of origin and the identification of the succession of dots to the inclusive succession of segments separating these dots from the origin (Vergnaud and Errecalde, 1980). Some 14–15 year-old students keep seeing the dots as ordered whole numbers 1, 2, 3, 4... from left to right, without being able to give any meaning to the interval between 2 and 3. When the origin cannot be represented, or when students have to change the origin of the reference system, many of them just fail. This conceptual difficulty is especially important in the use of graphic computer aids, when the origin and the scale are changed to focus some part of the graph (Nadot, 1988).

Linguistic and non-linguistic symbols are both a help and a problem for students. They help students in identifying the relevant mathematical objects and relationships, but they also raise problems of reading and understanding.

Psychology, epistemology and didactics

Research on Mathematics Education is a complex story. Psychology is only part of it, but has an essential part to play in it, from a theoretical and from a methodological point of view; not only to understand the different steps of the development of mathematical ideas and competences in students' minds, and the difficulties and the errors (as I have tried to show); but also to understand the process of learning in the classroom and the process of teaching.

Teaching and learning in the classroom is a social process, that depends on some macrofeatures of the educational and social system, and on microphenomena that take place in the interaction of students with mathematical situations and with the other actors (teacher and other students). The curriculum, the school books, the social environment, the system of training for teachers are macrophenomena. Chevallard (1985) has analysed the social process of transposition that governs the transformation (and deformation) of the scientific knowledge of mathematicians into the knowledge to be taught, and the knowledge actually taught. Among the factors of that process, teachers' representations, abilities, and difficulties play a very important part.

Three kinds of representations appear to be essential

- representation of the knowledge to be taught;
- representation of the students' competences and conceptions;
- representation of the learning process.

Brousseau has analysed different aspects of the erroneous representations of teachers concerning mathematics and the learning process. He has also developed the theory of didactic situations that concerns the invention, the choice and the management of situations to be offered to students. The theory of situations stresses the epistemological ground of the concept to be taught and proposes systematic ways of organizing students' activity and cooperation. He has for instance made the important distinction between situations aimed at producing an action, or a message or a proof. Brousseau has also developed the concept of "didactical contract" which implicitly rules the interaction of the students and the teacher.

One of the most important parameters of teachers' decisions in the choice of situations and in the teaching process is their representation of students' competences and conceptions. This is why, as a cognitive psychologist, I have stressed that point in this address. The emergence of competences can and must be described in mathematical terms, as "theorems-in-action". But the analysis of these competences requires more than the stabilized formal description that is usually offered by mathematics.

I have shown for instance that the analysis of additive structures requires change over time to be taken into account, also the use of a unary-operation model, also of children's own untaught procedures and errors. Mathematics does not usually take change over time into consideration, and sees addition as an internal binary law of combination. Actually students have to do with both a unary and a binary conception of addition, depending on the situations they have to master. There lies the conceptual problem.

Along the same idea, the analysis of multiplicative structures requires dimensional analysis to be taken into account, even at the primary school level. This means that the learning, and therefore the teaching of mathematics are not concerned only with pure numbers but also with magnitudes and relationships between magnitudes including quotients and products of dimensions. This analysis of additive and multiplicative structures might have been made from an *a priori* point of view. It just happens that it is the difficulties of students, their errors and procedures that have compelled us to revise and complete our views, *a posteriori*.

Cognitive psychology is essential to epistemology of mathematics as much as epistemology of mathematics is essential to cognitive psychology.

If epistemology is concerned with the relationship of concepts and procedures with the practical and theoretical problems to be solved, then epistemology is important not only to understand the initial competences and conceptions of children in a conceptual field, and the situations which shape these competences and conceptions, but also to understand how these competences and conceptions are enriched, widened, restricted and sometimes profoundly changed during the process of learning and during the cognitive development of students. This process covers many years. Some aspects of the concept of volume are grasped by 6 or 7 year olds: comparing the quantity of orange juice in glasses, and measuring and comparing the capacity of containers. But it is another story to grasp a tridimensional conception of volume; most 13 or 14 year olds do not satisfy the criterion of combining ratios on lengths, areas and volumes, as I have shown above.

My final conclusion is that didactics, psychology and epistemology must be strongly tied together in mathematics education research.

Epistemology is essential to imagine didactic situations and to characterize in mathematical terms students' competences and conceptions.

Psychology is essential to analyse carefully the short term process taking place in the classroom both from a cognitive point of view and from the point of view of social interaction; also to analyse the long term process of development.

Didactics includes epistemology and psychology, but it also takes it as a burden to theorize about the conception of situations and activities that should take place in the classroom and the way to manage the ensuing process.

A mathematical concept, if one looks at it developing in students' minds, is a triplet of three sets:

– the set of situations that make the concept meaningful in a variety of aspects;

- the set of operational invariants (properties, relationships, objects, theorems-in-action...) that are progressively grasped by students, in a hierarchical fashion;
- the set of linguistic and non-linguistic symbols that represent those invariants and are used to point at them, to communicate and discuss about them, and therefore to represent situations and procedures.

The symbolic dimension of mathematical concepts is essential, but we must never forget that mathematics is not a language but knowledge; solving practical and theoretical problems is both the source and the criterion of mathematical knowledge for students as much as for mathematicians.

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